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XVIII.—*The Amplitude of Vibration of Ions in the Crystals* NaCl, NaF, LiF, *and* KCl.

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(1) *Introduction.*

THE intensity with which a beam of X-rays is reflected from a crystal depends on many factors, but of these two are of special importance, namely, the distribution of charge in the atoms or ions † composing the crystal and their thermal agitation. The amplitude of a reflected X-ray beam is a function of the angle of scattering, 2θ, θ being the glancing angle of incidence, and of the wavelength, λ. Considered classically, the decrease of the reflected amplitude as θ increases is due to interference occurring between waves scattered from different parts of the vibrating atoms. Now, if an atom in a crystal can be regarded as a spatial distribution of charge surrounding a nucleus which vibrates as a whole or almost so—and evidence tends to show that this is approximately true [1], then the effect of the thermal agitation may be distinguished from the effect of the space charge distribution, and the two be considered separately.

* Communicated by Prof. R. Whiddington, F.R.S.

† Hereinafter "atoms" will include "ions."

The effect of the space charge distribution is conveniently expressed by F_0, a function of θ and λ, which has been defined [2] as the ratio of the coherent radiation scattered by an atom *at rest* in a direction θ to the amplitude which would be scattered in the same direction according to the classical theory by the same number of electrons as the atom contains considered as coincident point charges. This definition leads to a simple expression for F_0 for atoms having spherical symmetry :—

$$F_0 = \int_0^\infty U(r) \frac{\sin \phi}{\phi} . dr, \quad . \quad . \quad . \quad . \quad (1)$$

where $$\phi = \frac{4\pi}{\lambda} . r \sin \theta,$$

$U(r)$ being the radial density of charge at distance r from the nucleus, expressed in terms of the electron as unit. In applying equation (1) to atoms in a crystal a number of assumptions are made, some of which it is worth while to discuss briefly.

Firstly, the assumption that the classical formula for the scattering of X-rays by a free electron due to J. J. Thomson is applicable to the electrons in an atom has been considered by I. Waller in two papers, in the first of which [3] he shows that for an atom having one electron in a field of charge Ze the classical formula is still approximately valid from the point of view of quantum mechanics, and in the second [4] he extends this result to a many-electron atom, the condition to be satisfied being that the frequency of the radiation must be essentially higher than the K-absorption frequency of the scattering atom. This question is also discussed in a later paper by Waller and Hartree [5] on the total scattering of X-rays.

Secondly, it is generally assumed in comparing F curves obtained from experimental data with theoretical F curves based on calculated charge distributions that atoms in crystals are approximately spherical in shape, so that equation (1) can be applied to the theoretical charge distribution, $U(r)$. The justification for this is that any lack of spherical symmetry will be mainly confined to the outer parts of the atoms where $U(r)$ is small and, r being large, $(\sin \phi)/\phi$ also small—that is to say, deviations from spherical symmetry will be appreciable only in those parts of an atom which make very small contributions to the F

function. This is very clearly shown by a number of curves given by James, Waller, and Hartree [2] for the function $U(r) \sin \phi / \phi$ for Cl^- for different values of the parameter $(\sin \theta)/\lambda$. For crystals of the rock-salt type considered in this paper the assumption is certainly justified, for the spacing of the atoms in these crystals is so large that there is very little "overlapping" or "interpenetration" of neighbouring atoms; this is seen in fig. 3.

The effect of the thermal vibration of atoms in crystals on the amplitude of a reflected X-ray beam has been considered by Debye [6], and also by Waller [6], who has modified slightly the expression originally given by Debye. Experiments by James, Firth [6] and the present writer [7] on the reflexion of X-rays from rock-salt and sylvine over a wide range of temperature have shown that Waller's modification of Debye's formula represents very closely the variation of the intensity of reflexion with temperature from the temperature of liquid air, 86° Abs., to about 500° Abs. The Debye-Waller formula may be expressed as follows[8]:—

$$F_T = F_0 e^{-M}, \quad \ldots \ldots \ldots \quad (2)$$

where

$$M = 8\pi^2 . \overline{u_x^2} \left(\frac{\sin \theta}{\lambda} \right)^2,$$

$\overline{u_x^2}$ being the mean square displacement of an atom in an arbitrary direction x. For a crystal of the rock-salt type, $\overline{u^2} = 3\overline{u_x^2}$, $\overline{u^2}$ being the mean square of the total displacement. Hence (2) becomes

$$F_T = F_0 e^{-\alpha(\sin^2 \theta / \lambda^2)}, \quad \ldots \ldots \quad (3)$$

where

$$\alpha = \frac{8\pi^2}{3} . \overline{u^2};$$

$\sqrt{\overline{u^2}}$ may be regarded as the mean amplitude of vibration at temperature T. F_T and F_0 are respectively values of the F function for an atom vibrating at temperature T and for an atom *at rest*.

F_T has been obtained experimentally for the following crystals of the rock-salt type:—NaCl and KCl by James, Firth [6], and Brindley [7], NaCl, NaF, and LiF by Havighurst [9]. F_0 has been calculated by means of equation (1) for the ions Na^+, K^+, Li^+, Cl^- and F^- composing these crystals from theoretical charge distributions. From this data, using equation (3), $\sqrt{\overline{u^2}}$ may be

derived for each ion in these crystals at the temperature T at which the experimental measurements were made, *i. e.*, about 290° Abs.

It seems desirable here to compare the present method of obtaining $\sqrt{\overline{u^2}}$ with that used by James and his co-workers. They make use of the following expression derived by Waller [9] :—

$$\overline{u_x^2} = \alpha + \beta T + \gamma/T + \delta/T^3 + , \quad . \quad . \quad . \quad . \quad (4)$$

in which α is zero if zero-point energy having Planck's value is assumed, and γ and δ can be calculated with sufficient accuracy. β is obtained *experimentally* by making measurements of F_T at two temperatures, 86° Abs. and 290° Abs. In this way $\sqrt{\overline{u^2}}$ is obtained without any reference to theoretical charge distributions. However, when F_0 is calculated from purely experimental data, it is found to agree quite closely with F_0 deduced from the charge distributions obtained theoretically by the method due to Hartree [10]. This result is of considerable importance because, apart from being powerful evidence for the existence of zero-point energy, it shows that F_0 may be calculated with a fair degree of certainty* from the theoretical charge distributions, and that the two assumptions described briefly above relating to the use of equation (1) are justified.

(2) *Theoretical Values of the Function* F_0.

The distributions of charge in the ions considered have been obtained by Hartree's method [10]. With the exception of F^-, these distributions had already been calculated, Li^+ †, Na^+, and Cl^- by Hartree [10], and K^+ by James and the writer [7]. The distribution in F^- has been worked out by the "self-consistent field" method, using values of the initial field estimated by Hartree, and the results are given below in Table I.

In Table I. r is given in terms of the atomic unit of length ($=a_H=0{\cdot}532$ Å.U.) and $U(r)$ is in electrons per atomic unit of length.

Values of F_0 obtained from these charge distributions are given in Table II.

* This point is considered again later.

† I am indebted to Prof. Hartree for permission to use his results for Li^+ previous to their publication, and also for an estimate of the "initial" field of F^- used in calculating the charge distribution.

TABLE I.

Radial Distribution of Charge in Fluorine$^-$ ion.

r in atomic units.	U(r).	r in atomic units.	U(r).
0	0	1·2	4·32
·02	1·58	1·4	3·37
·04	4·39	1·6	2·58
·06	6·99	1·8	1·98
·08	8·68	2·0	1·52
·10	9·59	2·2	1·18
·12	9·75	2·4	0·92
·14	9·41	2·6	0·73
·16	8·80	2·8	0·58
·18	7·99	3·0	0·46
·20	7·14		
		3·5	0·28
·25	5·34	4·0	0·17
·30	4·16	4·5	0·11
·35	3·75	5·0	0·07
·40	3·83	5·5	0·045
		6·0	0·03
·5	4·60	6·5	0·02
·6	5·30	7·0	0·01
7	5·81	7·5	0·005
·8	5·88	8·0	0·00
·9	5·67		
1·0	5·29		

TABLE II.

Theoretical Values of the Function F_c.

$\left(\frac{\sin\theta}{\lambda}\right).10^{-8}$.	Li^+.	F^-.	Na^+.	Cl^-.	K^+.
0	2·00	10·00	10·00	18·00	18·00
·05	1·99	9·49	9·87	17·11	17·72
·10	1·96	8·65	9·50	15·23	16·46
·15	1·88	7·72	8·92	13·19	14·82
·20	1·76	6·74	8·21	11·50	13·31
·25	1·64	5·70	7·45	10·23	12·00
·3	1·52	4·79	6·68	9·30	10·78
·4	1·28	3·54	5·23	8·06	8·83
·5	1·04	2·75	4·07	7·23	7·77
·6	0·82	2·20	3·22	6·49	7·05
·7	0·64	1·92	2·63	5·77	6·44
·8	0·50	1·72	2·23	5·06	5·90
·9	0·40	1·57	1·96	4·41	5·32
1·0	0·33	1·48	1·75	3·84	4·79
1·1	0·26	1·36	1·59	3·33	4·22

(3) *Calculation of the Amplitudes of Vibration.*

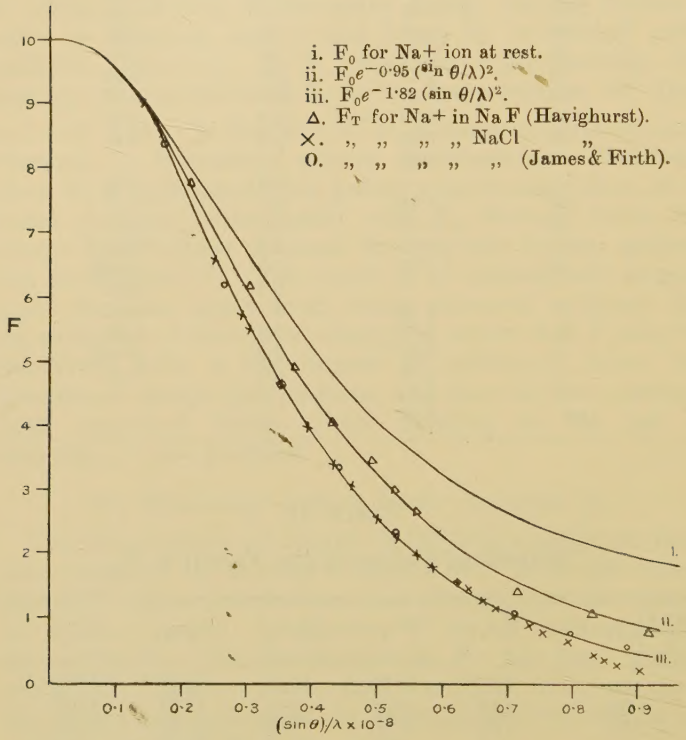
From equation (3)

$$\overline{u^2} = (3/8\pi^2)\alpha$$

and

$$\alpha = \frac{\log_e (F_0/F_T)}{(\sin\theta/\lambda)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Fig. 1.

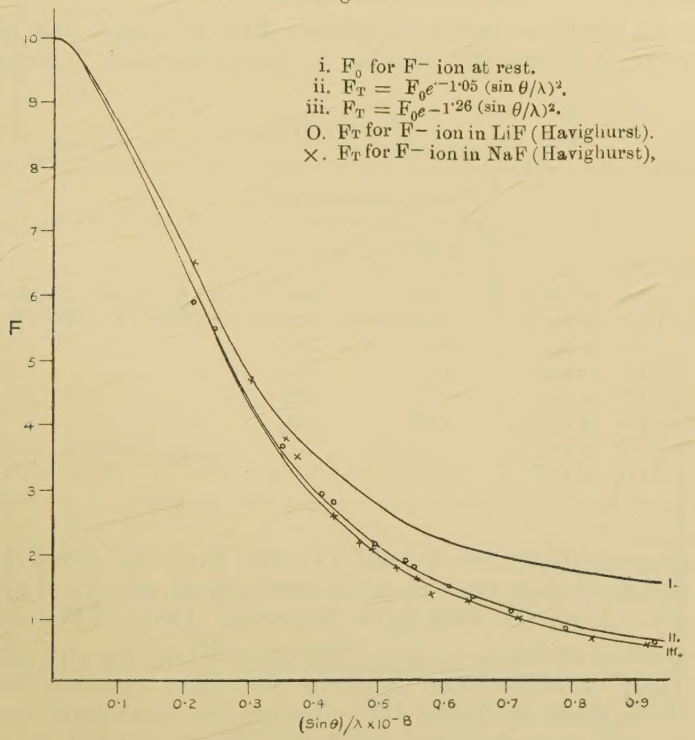

i. F_0 for Na+ ion at rest.
ii. $F_0 e^{-0.95 (\sin\theta/\lambda)^2}$.
iii. $F_0 e^{-1.82 (\sin\theta/\lambda)^2}$.
Δ. F_T for Na+ in Na F (Havighurst).
×. ,, ,, ,, ,, NaCl ,,
O. ,, ,, ,, ,, ,, (James & Firth).
F
(sin θ)/λ x 10⁻⁸
I.
II.
III.

F Curves for Na^+ ion.

A direct application of (5) to the F_0 and F_T curves for each ion enabled a series of values of α to be obtained for various values of $\sin\theta/\lambda$ which were found to be approximately constant. $\sqrt{\overline{u^2}}$ was then calculated from a mean value of α. Further, as a check, F_T was calculated from equation (3), using the mean value obtained for α, and compared with the experimental data.

In figs. 1 and 2 the results are shown for the Na^+ ion in NaCl and NaF and for the F^- ion in NaF and LiF. For the Na^+ ion in NaCl, the experimental results are shown obtained both by Havighurst and by James and Miss Firth. It is of interest to note that there is a close agreement between the two sets of experimental data for all values of $(\sin\theta)/\lambda$ except beyond 0·8, where the powder method used by Havighurst appears to give results which

Fig. 2.



F Curves for F^- ion.

are too small, while the single crystal method used by James and Miss Firth gives results in better agreement with the theoretical curve. For all the ions considered—with, perhaps, the exception of Li^+, for which it is difficult to get an experimental F curve—the values of F_T given by Havighurst for *small* values of $(\sin\theta)/\lambda$ fit quite well onto

the theoretical curves, a result which is in agreement with Havighurst's conclusion that the fine powder method eliminates the uncertainty at small angles due to secondary extinction. For average values of $(\sin\theta)/\lambda$, between 0·2 and 0·8, when α is rightly chosen there is a very close agreement between the experimental $\mathbf{F}_T$ values and the theoretical $F_0 e^{-\alpha(\sin^2\theta/\lambda^2)}$ curve.

In Table III. values are given of α and $\sqrt{\overline{u^2}}$ obtained from Havighurst's data and also values obtained by James [7] [9] using a different method.

In the experiments on sylvine, KCl, by James and the writer [7] it was not possible to obtain separate F curves

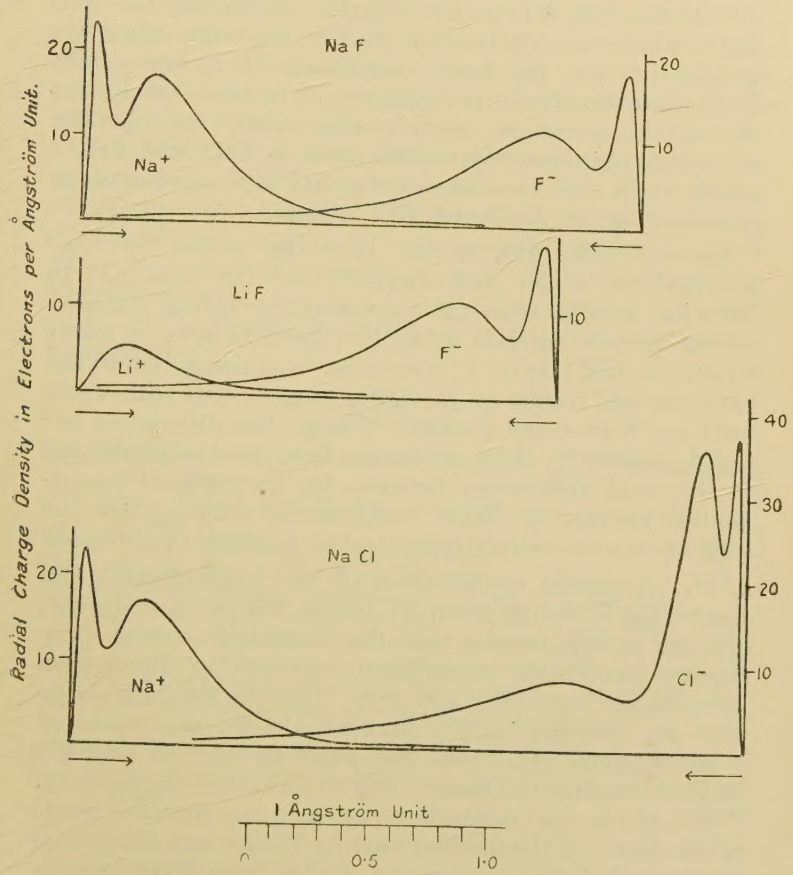
TABLE III.

Ion.	From Havighurst's data. α, in Å.U^2.	From Havighurst's data. $\sqrt{\overline{u^2}}$, in Å.U.	$\sqrt{\overline{u^2}}$, in Å.U.	
Na^+ in NaCl ...	1·82	$0·26_3$	0·242	Waller & James, *loc. cit.*
Cl^- in NaCl ...	1·60	$0·24_7$	0·217	
Na^+ in NaF ...	0·95	$0·19_0$		
F^- in NaF ...	1·26	$0·21_9$		
Li^+ in LiF.....	1·50	$0·23_9$		
F^- in LiF.....	1·05	$0·20_0$		
K^+ in KCl.....	—	—	0·255	James & Brindley, *loc. cit.*
Cl^- in KCl.....	—	—		

experimentally for K and Cl^-, for, since the diffracting powers of these two ions are so nearly equal, spectra of type [K—Cl] are too weak to be measured. Hence it was only possible to obtain a *mean* value of $\sqrt{\overline{u^2}}$ for the two ions. There is some data available for other crystals than those given in Table III. Results for aluminium have recently been published, by James, Brindley, and Wood [11]. Results have also been obtained for fluorite, CaF_2, by Havighurst and by James and Randall, and many results have also been obtained for several specimens of this crystal by the writer and R. G. Wood, but the investigation is not yet complete. It seems better therefore to restrict the present paper to crystals of the rock-salt type, and to leave the consideration of aluminium, fluorite, and possibly of other crystals to a later paper.

To enable the results obtained for $\sqrt{u^2}$ to be visualized easily in their relation to the sizes of the ions and the spacings of the crystal lattices fig. 3 was drawn. It shows the theoretical *radial* charge distributions for the ions (considered at rest) in NaF, LiF, and NaCl set at a

Fig. 3.



The arrows show the magnitude of the mean amplitudes of vibration of the ions at room-temperature.

distance apart equal to half the (1.0.0) spacings, *i. e.*, at the distance of closest approach. The arrows give the magnitude of the mean amplitudes of vibration at room-temperature.

It is evident that the thermal vibrations in the crystals considered are all approximately of the same magnitude—in fact, they are so nearly alike that it would be rash at present to attempt to attach any great significance to the differences between them. Since the "size" of an ion is so vague a quantity it is hardly worth while to consider a possible connexion between size and amplitude of vibration. But it is perhaps significant that in LiF and NaCl, where one ion is *much* smaller than the other, the smaller ion has the larger amplitude of vibration. In NaF, however, the reverse appears to be true, but in this crystal the sizes are more nearly equal. In LiF the vibrations are definitely smaller than in NaCl and KCl, a result which may be related to the LiF lattice being much smaller than the NaCl and KCl lattices.

Lastly, Table III. shows that the values of $\sqrt{u^2}$ obtained by Waller and James [9] for Na^+ and Cl^- in NaCl are smaller than the corresponding values obtained by the present method using Havighurst's data, a result which at first sight is somewhat surprising, since the experimental results of Havighurst agree well with those used by Waller and James. Though the differences are small absolutely, they are larger than can be attributed to the small differences between the two sets of experimental results. A little consideration shows that the differences arise mainly from the two methods of obtaining $\sqrt{u^2}$. A careful examination of the experimental and theoretical F curves given by James, Waller, and Hartree (*loc. cit.* p. 343) reveals that the theoretical curves come slightly above the experimental curves in the region between $\sin\theta/\lambda = 0{\cdot}4$ and $0{\cdot}8$. Thus in choosing α so that F_T and $F_0 e^{-\alpha(\sin^2\theta/\lambda^2)}$ fit most closely, the values of α so obtained are, from the point of view of James's method, slightly too large; this explains the discrepancy. Which of the two methods gives the more accurate result is not clear; in the method used by Waller and James it is assumed that the temperature factor determined experimentally between 86° Abs. and 290° Abs. holds between 0° Abs. and 86° Abs., while the present method assumes, among other things, that the process of obtaining F_0 from the theoretical charge distribution by equation (1) is justified. However, the results obtained by either method are probably consistent among themselves.

Finally, I would like to express my thanks to Prof. Hartree for estimating the initial field of the fluorine ion, F^-, and also to Mr. F. Tyler, B.Sc., for assistance in the calculation of the charge distribution for this ion.

(4) *Summary.*

A brief account is given of the dependence of the intensity of reflexion of X-rays on the thermal vibration and charge distribution of the scattering centres (atoms or ions) in a crystal. It is pointed out that the experimental work of James and others justifies the assumption that the charge distribution in an ion *at rest* in a crystal is given approximately by Hartree's method for the charge distribution in a free ion. By comparing X-ray scattering curves (F curves) obtained experimentally for a number of ions at room - temperature with the corresponding theoretical curves for the ions at rest an estimate has been made of the amplitude of vibration of the ions at room-temperature.

References.

(1) See papers by R. W. James and others, to which references are given below.
(2) James, Waller, and Hartree, Proc. Roy. Soc. A, cxviii. p. 334 (1928).
(3) I. Waller, Phil. Mag. iv. p. 1228 (1927).
(4) I. Waller, *Zeit. f. Phys.* li. p. 213 (1928).
(5) Waller and Hartree, Proc. Roy. Soc. A, cxxiv. p. 119 (1929).
(6) James and Firth, Proc. Roy. Soc. A, cxvii. p. 62 (1927). This paper contains references to the work of Debye and Waller on the effect of temperature on the intensity of X-ray reflexion.
(7) James and Brindley, Proc. Roy. Soc. A, cxxi. p. 155 (1928).
(8) Waller and James, Proc. Roy. Soc. A, cxvii. p. 214 (1927).
(9) R. J. Havighurst, Phys. Rev. xxviii. p. 869 (1926).
(10) D. R. Hartree, Proc. Camb. Phil. Soc. xxiv. pp. 89, 111 (1928).
(11) James, Brindley, and Wood, Proc. Roy. Soc. A, cxxv. p. 401 (1929).

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XIX. *A Note on the Scattering Power of the Carbon Atom in Diamond for X-Rays.* *By* G. W. Brindley, *M.Sc., Assistant Lecturer in Physics, University of Leeds* *.

A SHORT time ago the writer † calculated from an approximate model of the carbon atom its scattering power for X-rays. The results were expressed in terms of F, a function of θ and λ which measures the scattering power of an atom for radiation of wave-length λ in a direction θ. The theoretical F curve was found to differ considerably from an experimental F curve obtained by M. Ponte ‡ for the carbon atom in diamond. Recently an entirely new experimental investigation has been made by Miss A. H. Armstrong §, who obtains values of F markedly different from those of Ponte. The purpose of this note is to point our briefly that there is a moderately good agreement between Miss Armstrong's results and the theoretical results.

The method previously used for calculating F was as follows :—Schrödinger's equation for ψ for an electron in a central Coulomb field of charge $Z'e$ can be solved explicitly ||, and its radial charge density $U(r)=4\pi r^2 . \psi\bar{\psi}$ can be calculated. It was assumed that each electron in the carbon atom could be considered to be in a central Coulomb field, Z' being an effective nuclear charge equal to $(Z-s)$, Z being the actual nuclear charge, 6, and s a screening constant.

Reasons were given for considering s to be small for the K electrons of carbon and of the order of 2 for the L electrons. $U(r)$ was calculated for a 1_1 electron with $Z'=6{\cdot}0$, and for a 2_1 and a 2_2 electron with $Z'=4{\cdot}0$. Then, using the equation ¶ :—

$$F=\int_0^\infty U(r)\frac{\sin\phi}{\phi}dr, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $$\phi=\frac{4\pi}{\lambda}r . \sin\theta,$$

F was calculated for each of these electrons.

* Communicated by Prof. R. Whiddington, F.R.S.

† G. W. Brindley, Proc. Leeds Phil. Soc. i. p. 402 (1929).

‡ M. Ponte, Phil. Mag. iii. p. 195 (1927); iv. p. 232 (1927).

§ A. H. Armstrong, Phys. Rev. xxxiv. p. 1115 (Oct. 1929).

|| E. Schrödinger, *Ann. d. Phys.* lxxix. p. 361 (1926).

¶ *Vide* James, Waller, and Hartree, Proc. Roy. Soc. A, cxviii. p. 334 (1928).

Despite the roughness of the method, one important conclusion was obtained, namely, that *for values of sin* $\theta > 0{\cdot}3$ *the contribution of the L electrons to the F curve for carbon is negligibly small.* This is clearly seen in the text-figure.

Now the K electron distribution in carbon will be determined mainly by the central nucleus, and the influence of the L electrons will be comparatively small; thus the K distribution will be spherically symmetrical, or almost so, and the field will be practically of the Coulomb type, so that the assumptions involved in the simple theory given above are justified for the K electrons. From the text-fig. it is seen that with only two exceptions all the experimental values of F occur at values of $\sin\theta > 0{\cdot}3$, so that in comparing the theoretical curve with the experimental results it is only the K electrons which are involved. The L electrons, the charge distribution of which is so doubtful, only contribute appreciably to the F curve at small values of $\sin\theta$ where there are practically no experimental data.

In the previous paper Z' was taken equal to 6·0 for the K electrons mainly for reasons of convenience; it was quite obvious that a small change of Z' for the K electrons could not possibly explain the low values of F obtained by Ponte. Since Miss Armstrong's values of F are considerably higher than Ponte's, it seems worth while to reconsider the value of s for the K electrons. A value 0·4 has been chosen, based on the following considerations:— the most accurate method which we have at present for calculating the charge distribution in a many-electron atom is that given by Hartree *. By comparing charge distributions obtained by Hartree's method with hydrogen-like distributions, Z' can be found. For K electrons $s = (Z - Z')$ does not vary rapidly with Z, and may therefore be obtained fairly accurately for any atom by interpolation. A small error in s of the order of 0·05 will have no appreciable effect on the F curve.

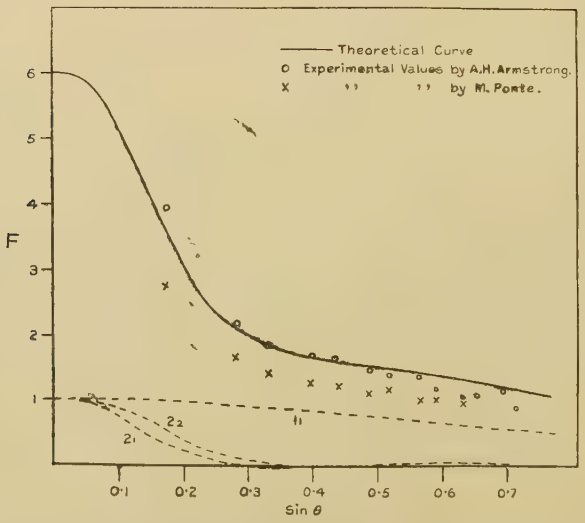
In the above table values are given of F for a 1_1 electron with $Z' = 5{\cdot}6$, and for a 2_1 and a 2_2 electron with $Z' = 4{\cdot}0$. These results are shown graphically in the text-fig. The full-line curve is the theoretical F curve for the carbon atom (assuming two 2_1 and two 2_2 electrons), the corresponding values being given in column five of the table. The circles

* D. R. Hartree, Proc. Camb. Phil. Soc. xxiv. pp. 89, 111 (1928).

show the experimental values obtained by Miss Armstrong, and the crosses the values obtained by Ponte.

Calculated Values of F.

Sin θ for λ=0·71 Å.U.	Values of F. 1_1 Z′=5·6.	2_1 Z′=4·0.	2_2 Z′=4·0.	F for Carbon Atom.
0·000	1·00	1·00	1·00	6·00
0·066	0·99	0·91	0·97	5·74
0·133	0·98	0·50	0·73	4·42
0·199	0·95	0·21	0·38	3·08
0·267	0·91	0·03	0·16	2·20
0·332	0·87	—0·01	0·06	1·84
0·400	0·81	—0·01	0·01	1·62
0·467	0·76	0·01	0·00	1·54
0·534	0·70	0·03	—0·01	1·44
0·667	0·58	0·04	—0·01	1·22
0·800	0·47	0·03	—0·01	0·98



The agreement between the calculated curve and the more recent experimental values is as good as can be expected. At large values of sin θ the experimental values tend to fall below the theoretical curve. A number of reasons may explain this. Firstly, any vibration of the

atoms in the lattice will affect the higher order spectra much more than the lower order spectra. The effect of atomic vibration on the scattering of X-rays has been considered in particular by James and his co-workers and by Waller*. If F_0 is the F function for an atom at rest, then we can write

$$F = F_0 e^{-M},$$

where

$$M = 8\pi^2 \sin^2\theta \,.\, \overline{u_x^2}/\lambda^2,$$

where $\overline{u_x^2}$ is the mean square displacement in an arbitrary direction x. In diamond the atoms are very closely packed. The hardness of the crystal, its very high characteristic temperature †, 1830° C., and other physical properties support the view that the atomic vibrations in diamond are very small. Further, experiments on the reflexion of X-rays over a wide range of temperature have failed to detect any change of intensity with temperature ‡. All evidence tends to show, therefore, that the atomic vibrations in diamond are very small, and if they exist at all that they are probably not large enough to explain the observed difference.

A second possible explanation is that measurements of the scattering factor F using powdered crystals appear to give results which are low at large values of sin θ compared with theoretical results, and also low compared with results from single crystals. This is referred to in the preceding paper by the writer on the amplitude of vibration of ions in a number of crystals of the rock-salt type.

Another possible explanation, but one which it is difficult to consider quantitatively, is this : the existence of the (222) spectrum is strong evidence for the distortion of the L electron shell of the carbon atom in diamond, and it may be that between the atomic planes there is a sufficient density of charge to scatter radiation of an appreciable amount out of phase with that scattered by the charge near the planes §. The irregularity of the F values at large sin θ, if not due to experimental causes, is in agreement qualitatively with this point of view.

* *Vide* Waller and James, Proc. Roy. Soc. A, cxvii. p. 214 (1927), where references are given to other work.

† *Cf.* P. Debye, *Ann. d. Phys.* xliii. p. 88 (1914).

‡ I. Backhurst, Proc. Roy. Soc. A, cii. p. 340 (1922). Ehrenberg, Ewald, and Mark, *Zeit. f. Krist.* lxvi. p. 547 (1928).

§ *Cf.* W. H. Bragg, Proc. Lond. Phys. Soc. xxxiii. p. 304 (1921).

To summarize : a theoretical F curve is calculated for carbon which, beyond sin $\theta = 0{\cdot}3$, depends almost entirely on the K electrons for which the simple theory is adequate. There is a satisfactory agreement between the calculated curve and recent experimental data. Possible explanations are suggested for the tendency of the experimental values to fall slightly below the curve at large values of sin θ.

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Dec. 10th, 1929.

XX. *On the Principle of Inaccessibility of the Absolute Zero.* *By* N. A. Kolossowsky, *D.Sc., Professor of Physical Chemistry and of Thermodynamics, University of Leningrad* *.

ONE generally attributes the first idea concerning the principle of inaccessibility of the temperature of the absolute zero to Nernst †, who in a paper published in 1912, had enounced this principle and, in addition, had attempted to intimately bind it with his new heat-theorem ‡. Nevertheless, this opinion does not in any way correspond with the historic reality.

Indeed, E. Hoppe has recently indicated in his 'History of Physics' §, that analogical ideas had been enunciated already in 1874 by Koppe ‖. But Hoppe is incorrect in

* Communicated by the Author.

† See, *e. g.*, P. Pawlowitsch, *Journ. Russ. Phys.-Chem. Soc.*, Phys. pt. xlv. (2) p 253 (1913). H. A. Lorentz, *Chem. Weekbl.* x. p. 621 (1913); and *Journ. Russ. Phys.-Chem. Soc.*, Phys. pt. xlvi. p. 4 (1914). P. Czukor, *Verhandl. deut. phys. Ges.* xvi. p. 486 (1914); M. Polanyi, *Verhandl. deut. phys. Ges.* xvi. p. 333 (1914), and xvii. p. 350 (1915). A. Einstein, 'Rapports et discussions du 2me Conseil de physique de l'Institut, Solvay,' Paris, pp. 293–298. Article of K. Bennewitz in the *Handbuch der Physik*, herausgegeb von Geiger und Scheel, ix. p. 172 (1926). W. Meissner, *ZS. f. Phys.* xxxvi. p. 325 (1926). A. Schidlof, *Journ. chim. phys.* xxiii. p. 814 (1926). W. Jazyna, *ZS. f. Phys.* xli. p. 211 (1927). F. Simon, *ZS. f. Phys.* xli. p. 806 (1927). H. Mache, *Ber. Wien. Akad. Wiss.* Abt. II *a.* cxxxvi. p. 75 (1927).

‡ W. Nernst, *Ber. Preuss. Akad. Wiss.* S. 134 (1912); see also W. Nernst, 'Die theoretischen und experimentellen Grundlagen des neuen Wärmesatzes,' 2te Aufl. S. 72 (1924).

§ E. Hoppe, 'Geschichte der Physik. Braunschweig' (1926) (Kinetische Gastheorie).

‖ Koppe, Pogg. *Ann.* cli. p. 643 (1874).

stating that Koppe had been the pioneer in this way, for in reality already in 1862 R. Clausius * had given the first formulation of this principle which was expressed by him in a perfectly clear manner and without any ambiguity. It is strange enough that nobody has up to date mentioned this circumstance, so much the more that, if one occupies oneself in researches in the domain of these questions, a deep study of Clausius's classical works is absolutely indispensable.

Clausius even gives a demonstration of the inaccessibility of the absolute zero, obviously deriving from the second law and particularly from the following formula :

$$Z - Z_0 = m \cdot c \cdot \ln \cdot \frac{T_0}{T},$$

where m represents the mass, c the true heat capacity †, and finally $Z - Z_0$ the variation of disgregation ‡, *i. e.* of the degree of division of the substance corresponding to a change of temperature from T_0 to T. This formula allowed Clausius to draw the following inference. "If one should wish to bring a body to the absolute zero of temperature the change of disgregation would become infinitely great, because in this case one would put $T = 0$ into the foregoing formula. This proves on principle that it is impossible to produce by means of any changes of state of a body such a cold that one could attain the absolute zero of temperature."

One obviously may discuss the objective value of Clausius's demonstration, as well as of all the other more recent ones, concerning the inaccessibility of the absolute zero, but at any rate, as regards the idea itself of this principle, one is indubitably obliged to agree, that in Clausius's statement it is as clear and precise as in the well-known one of Nernst.

* R. Clausius, *Vierteljahrschrift Ges. Naturwiss. Zürich*, vii. p. 48 (1862); Pogg. *Ann.* cxvi. p. 73 (1862); Phil. Mag. (4) xxiv. pp. 81, 201 (1862); *Journ. de Liouville* (2) vii. p. 209 (1862); 'Théorie mécanique de la chaleur,' trad. par F. Folie, Paris, Lacroix, Première partie, pp. 291–293 (1868–69).

† It is here the question of heat capacities c and c' defined by the equations

$$c = c_v - AT \int \left(\frac{\partial^2 p}{\partial T^2} \right) dv \quad \text{and} \quad c' = c_p + AT \int \left(\frac{\partial^2 v}{\partial T^2} \right)_p dp.$$

These heat capacities are independent of the temperature, as well as of the state of aggregation of the substance.

‡ Conception, having a certain analogy with the modern conception introduced into the science by G. N. Lewis under the denomination of fugacity.

It is precisely to this historic fact that we wish to draw the attention of physicists, as it incontestably refers to a very important and actual problem.

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XXI. *Effect of a Circular Hole on the Stress Distribution in a Beam under Uniform Bending Moment.* *By* Zirô Tuzi, *Research Fellow, the Institute of Physical and Chemical Research, Tokyô* *.

[Plate II.]

I. Introduction.

THE effect of a circular hole on the stress distribution in a plate under uniform *tension* has been studied by several authors both theoretically and experimentally; on the other hand, the case of *bending*, which is more important for various practical purposes, has been but little studied †, so far as the author is aware. The object of the present paper is to give full details of the experimental investigation of the stress distribution in a beam having a circular hole of various diameters on the neutral axis and subjected to a uniform bending moment, and to compare the results with the theoretical solution, which is also worked out in this paper. The experimental results are obtained by means of photo-elasticity, using a new material, "Phenolite" ‡, which is optically five times more sensitive than xylonite; and the values of stresses are measured directly, counting the numbers of interference fringes secured on photographic plates by means of monochromatic light 5461 Å. from an intense mercury lamp, so that practically no personal error can possibly enter. A uniform bending moment is applied on a beam AB, as shown in fig. 1, and photographs of its

* Communicated by Mr. Uzumi Doi.

† E. G. Coker, 'Engineering,' March 8th, 1912. T. Fukuhara, Journ. Soc. Mech. Engr. (Tokyo), xxxi. no. 133, p. 169 (1928).

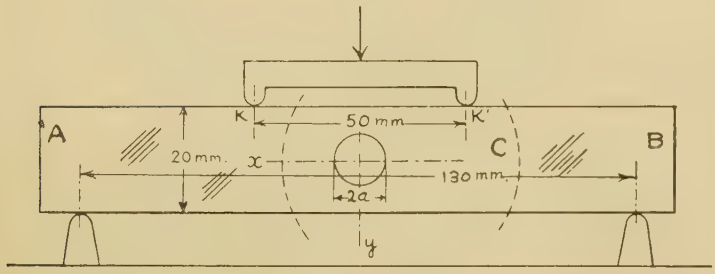
‡ Z. Tuzi, Sci. Pap. Inst. Phys. Chem. Res. (Tokyo), vii. nos. 112–114, pp. 79–120 (1927).

central portion C are taken by the polarized light passing through it: a few of them are reproduced in Pl. II. There are so many fringes revealed sharp and clear, as will be seen in the reproduced examples, that it is easy to draw an accurate contour of stresses from the corresponding negative plates.

II. Theory.

From the photograph (Pl. II.), the tangential distribution along the periphery of the hole is obtained, as shown in fig. 2. The diagram suggests that the stress function is likely an odd one, and the following solution is worked out, for which the author wishes to express his indebtedness to Prof. N. Yamaguti, whose kind guidance facilitated the

Fig. 1.



solution. In the solution the plate of the specimen is assumed to extend infinitely, and a circular hole of the radius a is made with its centre falling on an origin arbitrarily taken through which axes of x and y are drawn. It is also assumed that the uniform bending-moment is so applied that everywhere the stress distributions are given, in the case where there is no hole, by $\widehat{xx} = Ay$ and $\widehat{yy} = \widehat{xy} = 0$, in which $\widehat{xx}$ and $\widehat{yy}$ are the normal stresses, $\widehat{xy}$ the shear stress, and A is a constant.

Denoting by χ the stress function, and using the usual notations, in x-y coordinates, we have

$$\nabla^2\nabla^2\chi = 0, \qquad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$\widehat{xx} = \frac{\partial^2\chi}{\partial y^2}, \quad \widehat{yy} = \frac{\partial^2\chi}{\partial x^2}, \quad \widehat{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y}.$$

In polar coordinates

$$\nabla^2\nabla^2\chi = 0, \qquad \text{where } \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2.$$

The boundary conditions are given by

$$\widehat{xx} = Ay = Ar\sin\theta, \quad \widehat{yy} = 0, \quad \widehat{xy} = 0, \quad \text{at } x = \pm\infty; \quad \text{(I.)}$$

$$\widehat{rr} = 0, \quad \widehat{r\theta} = 0, \quad \text{at } r = a; \quad \text{(II.)}$$

where A and a are constants, and (I.) is satisfied by

$$\chi_0 = \frac{A}{6}y^3 = \frac{A}{6}r^3\sin^3\theta. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put $\chi = \chi_0 + \chi_1$, and we shall have

$$\widehat{rr} = \left(\frac{1}{r}\frac{\partial\chi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi_0}{\partial\theta^2}\right) + \left(\frac{1}{r}\frac{\partial\chi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi_1}{\partial\theta^2}\right),$$

$$\widehat{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi_0}{\partial\theta}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi_1}{\partial\theta}\right).$$

At $r = a$: $\widehat{rr} = 0$ and $\widehat{r\theta} = 0$.

$$\left.\begin{aligned} \therefore \quad \left[\frac{1}{r}\frac{\partial\chi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi_1}{\partial\theta^2}\right]_{r=a} &= -\left[\frac{1}{r}\frac{\partial\chi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi_0}{\partial\theta^2}\right]_{r=a} \\ &= -Aa(\sin\theta - \sin^3\theta), \\ \left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi_1}{\partial\theta}\right)\right]_{r=a} &= -\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi_0}{\partial\theta}\right)\right]_{r=a} \\ &= Aa(\cos\theta - \cos^3\theta). \end{aligned}\right\} \quad (2)$$

Assuming

$$\chi_1 = b_1 r^{-1}\sin\theta + (b_3 r^{-3} + d_3 r^{-1})\sin 3\theta, \quad . \quad . \quad (3)$$

gives

$$\left.\begin{aligned} [\widehat{rr}]_{\chi_1,\, r=a} &= -\left(\frac{2b_1}{a^3} + \frac{36b_3}{a^5} + \frac{30d_3}{a^3}\right)\sin\theta \\ &\quad + \left(\frac{48b_3}{a^5} + \frac{40d_3}{a^3}\right)\sin^3\theta, \\ [\widehat{r\theta}]_{\chi_1,\, r=a} &= \left(\frac{2b_1}{a^3} - \frac{36b_3}{a^5} - \frac{18d_3}{a^3}\right)\cos\theta \\ &\quad + \left(\frac{48b_3}{a^5} + \frac{24d_3}{a^3}\right)\cos^3\theta. \end{aligned}\right\} \quad (4)$$

From (2) and (4) :

$$\left.\begin{aligned} \frac{2b_1}{a^3}+\frac{36b_3}{a^5}+\frac{30d_3}{a^3} &= \mathrm{A}a, \\ \frac{48b_3}{a^5}+\frac{40d_3}{a^3} &= \mathrm{A}a, \\ \frac{2b_1}{a^3}-\frac{36b_3}{a^5}-\frac{18d_3}{a^3} &= \mathrm{A}a, \\ -\frac{48b_3}{a^5}-\frac{24d_3}{a^3} &= \mathrm{A}a. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

From (5) we obtain

$$b_1=\frac{\mathrm{A}a^4}{8}, \quad b_3=-\frac{\mathrm{A}a^6}{12}, \quad \text{and} \quad d_3=\frac{\mathrm{A}a^4}{8}. \quad . \quad (6)$$

$$\therefore \quad \chi=\chi_0+\chi_1$$

$$=\mathrm{A}a^3\left\{\left(\frac{a}{2r}-\frac{a^3}{4r^3}\right)\sin\theta+\left(\frac{r^3}{6a^3}-\frac{a}{2r}+\frac{a^3}{3r^3}\right)\sin^3\theta\right\}. \quad . \quad . \quad . \quad (7)$$

Thus the stresses in polar coordinates run as follows:—

$$\widehat{rr}=\mathrm{A}a^3\left\{\left(\frac{r}{a^3}-\frac{4a}{r^3}+\frac{3a^3}{r^5}\right)\sin\theta-\left(\frac{r}{a^3}-\frac{5a}{r^3}+\frac{4a^3}{r^5}\right)\sin^3\theta\right\},$$

$$\widehat{\theta\theta}=\mathrm{A}a^3\left\{\left(\frac{a}{r^3}-\frac{3a^3}{r^5}\right)\sin\theta+\left(\frac{r}{a^3}-\frac{a}{r^3}+\frac{4a}{r^5}\right)\sin^3\theta\right\},$$

$$\widehat{r\theta}=-\mathrm{A}a^3\left\{\left(\frac{r}{a^3}+\frac{2a}{r^3}-\frac{3a^3}{r^5}\right)\cos\theta-\left(\frac{r}{a^3}+\frac{3a}{r^3}-\frac{4a^3}{r^5}\right)\cos^3\theta\right\}.$$

Along the periphery of the hole where $r=a$:

$$\widehat{rr}=\widehat{r\theta}=0,$$

$$\widehat{\theta\theta}=2\mathrm{A}a\sin\theta(2\sin^2\theta-1),$$

$\widehat{\theta\theta}=0$ at $\theta=n\pi$ $(n=0, 1, 2, 3, \ldots)$,

and at $\theta=\pm\frac{\pi}{4}(2n+1)$;

$\widehat{\theta\theta}$ is maximum at $\theta=\frac{\pi}{2}(2n+1)$,

and at $\theta=n\pi\pm 24^{\circ}\cdot 1$.

Expressed in x-y coordinates,

$$\chi = \mathrm{A}a^3 \left\{ \left[\frac{a}{2(x^2+y^2)} - \frac{a^3}{4(x^2+y^2)^2} \right] y + \left[\frac{1}{6a^3} - \frac{a}{2(x^2+y^2)^2} + \frac{a^3}{3(x^2+y^2)^3} \right] y^3 \right\};$$

$$\left\{ \begin{aligned}
\widehat{xx} &= \mathrm{A}a^4 y \left\{ \frac{1}{a^4} - \frac{1}{(x^2+y^2)^2} \left[6 - \frac{5a^2}{x^2+y^2} \right] \right. \\
&\qquad + \frac{2}{(x^2+y^2)^3} \left[9 - \frac{10a^2}{x^2+y^2} \right] y^2 \\
&\qquad \left. + \frac{4}{(x^2+y^2)^4} \left[-3 + \frac{4a^2}{x^2+y^2} \right] y^4 \right\}, \\
\widehat{yy} &= \mathrm{A}a^4 y \left\{ \frac{1}{(x^2+y^2)^2} \left[-1 + \frac{a^2+4x^2}{x^2+y^2} - \frac{6a^2x^2}{(x^2+y^2)^2} \right] \right. \\
&\qquad \left. + \frac{2}{(x^2+y^2)^3} \left[1 - \frac{6x^2+a^2}{x^2+y^2} + \frac{8a^2x^2}{(x^2+y^2)^2} \right] y^2 \right\}, \\
\widehat{xy} &= -\mathrm{A}a^4 x \left\{ \frac{1}{(x^2+y^2)^2} \left[-1 + \frac{a^2}{x^2+y^2} \right] \right. \\
&\qquad + \frac{2}{(x^2+y^2)^3} \left[5 - \frac{6a^2}{x^2+y^2} \right] y^2 \\
&\qquad \left. - \frac{4}{(x^2+y^2)^4} \left[3 - \frac{4a^2}{x^2+y^2} \right] y^4 \right\}.
\end{aligned} \right.$$

At $x = 0$: $\widehat{xy} = 0$,

$$\widehat{xx} = \mathrm{A}\left(y + \frac{a^6}{y^5} \right), \quad \widehat{yy} = \mathrm{A}a^4 \left(\frac{1}{y^3} - \frac{a^2}{y^5} \right).$$

At $y = 0$: $\widehat{xx} = \widehat{yy} = 0$

and $\widehat{xy} = \mathrm{A}a^4 \left(\frac{1}{x^3} - \frac{a^2}{x^5} \right)$,

so that $\widehat{xy}$ is maximum at $x = \pm 1{\cdot}29\,a$.

The case when the centre of the hole is shifted from the neutral axis can readily be worked out by superposing these results with that of the well-known solution for uniform tension.

Experiments.

The test beams of phenolite were of 20 mm. breadth and 6 mm. thickness, with the span-length of 130 mm., each having a circular hole of diameter 0 mm., 3 mm., 5 mm., 8 mm., 10 mm., and 12 mm. (specimen *a*, *b*, *c*, *d*, *e*, *f*) respectively. At the centre of the two symmetrical loading points K, K′ in fig. 1, 37·88 kg. weight was applied, that is, on each point K, K′ 18·94 kg., for which it was ascertained that even with the weakest specimen, with the hole of the diameter $2a=12$ mm., the maximum fibre stress-intensity did not exceed the elastic limit of the material. All the specimens had been annealed under quite the same conditions to eliminate the initial stresses, and the free boundaries, straight or circular, were filed off carefully just before the experiment, so as to attain the uniform darkness when illuminated without load. The photographs were taken* immediately after the load was applied, and the exposure of $\frac{1}{5}$ second was quite enough with a mercury green filter, so that they were free from the time-effect of the optical creep. As they are seen in Pl. II., sufficient numbers of sharp and clear fringes were displayed, and they are not only permanent and sure records, but are affected with little personal error indeed. The so-called "compensator," which was a small tension test-piece made from the same material as the main specimen, was used simply for the purpose of ascertaining the sense of the stresses.

First of all, the tangential stress-distributions along the periphery of the hole, where $r=a$, were thus obtained, and are shown in fig. 2, *f*, *e*, *d*, and *c*, and also in Table I. In the figures broken lines are drawn in accordance with the theoretical values calculated by the equations above mentioned. The value of the constant A was obtained from the photograph of the specimen *a*, in which $r=0$ mm., and also verified with another specimen at sections away from the hole so as to be free from its effect, and the value was found to be $A=1{\cdot}075\,\frac{\text{fringe order}}{\text{mm.}}$. In solving the differential equations, we assumed that the plate of the specimen extended infinitely, so that both the experimental

* The description of the experimental method is abridged in the present paper as it is quite the same as mentioned in the preceding paper of the author: "Photographic and Kinematographic Method for Photo-elasticity," Sci. Pap. Inst. Phys. Chem. Res. (Tokyo), viii. no. 149, pp. 247–267 (1928).

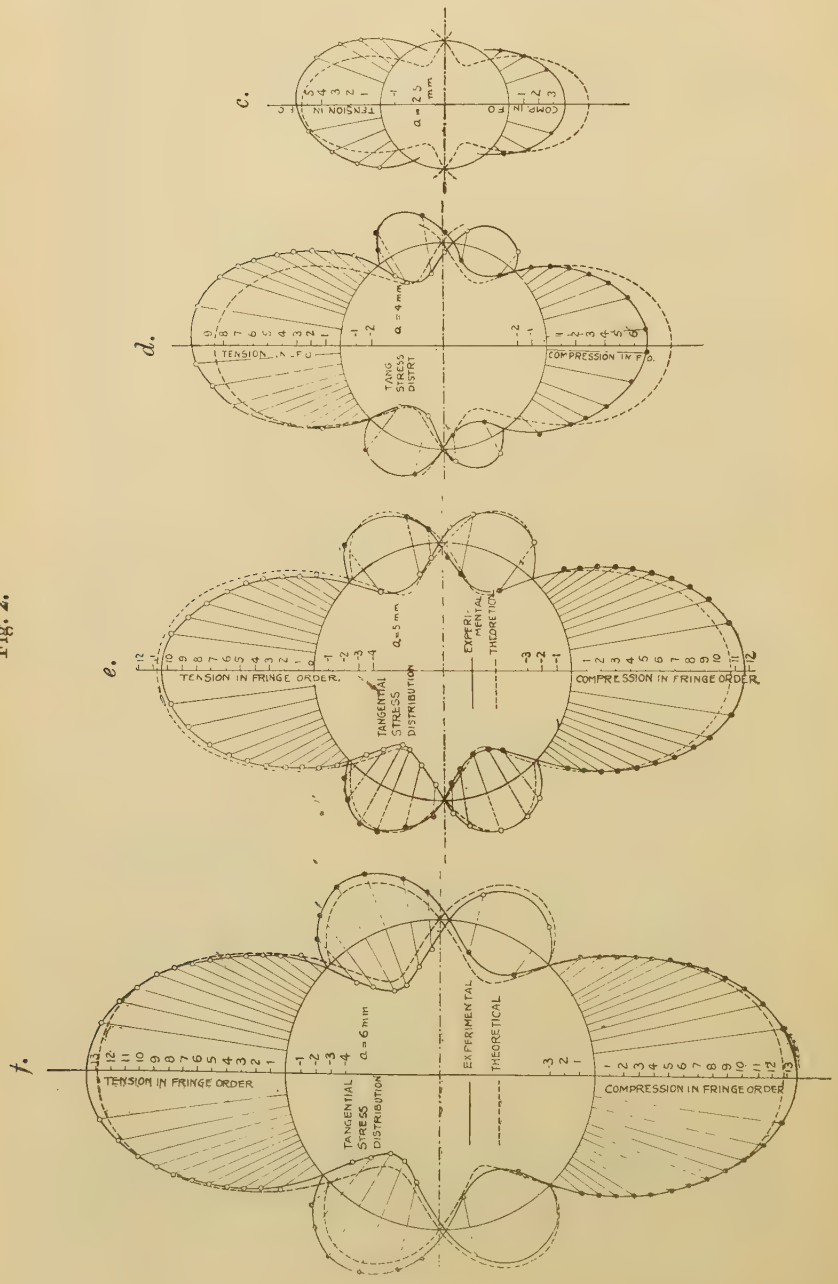

c.
a = 2.5 mm
TENSION IN FRINGE ORDER
COMPRESSION IN FRINGE ORDER
d.
a = 4 mm
TANG STRESS DISTRI
TENSION IN FRINGE ORDER
COMPRESSION IN FRINGE ORDER
e.
a = 5 mm
EXPERIMENTAL
THEORETICAL
TANGENTIAL STRESS DISTRIBUTION
TENSION IN FRINGE ORDER.
COMPRESSION IN FRINGE ORDER
f.
a = 6 mm
EXPERIMENTAL
THEORETICAL
TANGENTIAL STRESS DISTRIBUTION
TENSION IN FRINGE ORDER
COMPRESSION IN FRINGE ORDER

Fig. 2.

Remark.—In the following tables stresses are given in fringe order (f. o.).

TABLE I.

Specimen *f*. $a = 6$ mm.

First quadrant.		Second quadrant.		Third quadrant.		Fourth quadrant.	
$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.
−2	4·0	−2	175·0	2	−168·5	2	−13·2
−3	11·0	−3	168·0	2	−140·0	2	−36·2
−4	20·5	−4	158·0	− 2	−129·5	− 2	−49·7
−4	29·0	−4	148·8	− 3	−126·5	− 3	−53·0
−3	37·2	−3	141·2	− 4	−124·0	− 4	−56·0
−2	42·3	−2	135·8	− 5	−121·4	− 5	−58·5
2	49·6	3	126·4	− 6	−119·0	− 6	−60·8
3	53·7	4	123·0	− 7	−116·2	− 7	−62·9
4	56·4	5	120·7	− 8	−113·4	− 8	−65·6
5	59·2	6	118·2	− 9	−110·7	− 9	−67·8
6	61·3	7	116·0	−10	−112·8	−10	−70·8
7	64·0	8	113·6	−11	−105·0	−11	−73·4
8	66·5	9	111·2	−12	−101·6	−12	−77·0
9	68·8	10	108·5	−13	− 97·4	−13	−81·6
10	71·2	11	105·7				
11	74·7	12	102·2				
12	77·8	13	98·0				
13	81·8						

Specimen *e*. $a = 5$ mm.

First quadrant.		Second quadrant.		Third quadrant.		Fourth quadrant.	
$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.
−1	5·8	−1	175·4	1	−176·8	1	− 2·8
−2	13·2	−2	165·8	2	−171·2	2	−11·6
−2	38·2	−3	157·0	3	−161·0	2	−36·2
2	52·8	−3	150·6	3	−150·8	− 2	−50·4
3	56·8	−2	142·6	2	−143·4	− 3	−53·7

TABLE I. (*cont.*).

Specimen *e*. $a=5$ mm. (*cont.*).

First quadrant.		Second quadrant.		Third quadrant.		Fourth quadrant.	
$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.
4	59·7	−1	136·7	− 2	−128·7	− 4	−56·4
5	63·0	1	131·0	− 3	−125·8	− 5	−59·3
6	66·0	2	127·4	− 4	−123·0	− 6	−62·4
7	70·0	3	123·7	− 5	−120·3	− 7	−65·4
8	74·2	4	121·0	− 6	−117·0	− 8	−68·5
9	78·2	5	117·7	− 7	−114·0	− 9	−72·0
10	82·2	6	114·3	− 8	−111·0	−10	−75·8
		7	111·2	− 9	−112·5	−11	−81·0
		8	107·3	−10	−104·5		
		9	102·2	−11	− 99·3		
		10	96·7				

Specimen *d*. $a=4$ mm.

First quadrant.		Second quadrant.		Third quadrant.		Fourth quadrant.	
$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.
−2	10·0	−2	167·8	1	−174·5	−1	− 1·7
−2	31·0	−2	143·6	1	−152·4	1	−12·4
−1	34·3	2	129·0	−2	−132·5	1	−39·5
2	44·8	3	124·3	−3	−121·6	−1	−49·0
3	49·7	4	120·4	−4	−117·0	−2	−54·2
4	53·8	5	116·8	−5	−110·6	−3	−58·8
5	57·7	6	112·5	−6	−101·0	−4	−64·0
6	61·0	7	109·6	−7	− 92·0	−5	−70·0
7	65·8	8	105·7			−6	−75·8
8	68·8	9	100·0			−7	−85·5
9	73·0	10	94·5				
10	81·0						

TABLE I. (*cont.*).

Specimen *c*. $a=2{\cdot}5$ mm.

First quadrant.		Second quadrant.		Third quadrant.		Fourth quadrant.	
$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.	$\widehat{\theta\theta}$ in f. o.	Angle in degrees.
2	43·0	2	133·5	−1	−129·6	−1	−45·6
3	51·8	3	123·2	−2	−118·3	−2	−58·0
4	59·2	4	112·2	−3	−104·8	−3	−70·8
5	69·3	5	100·8				

Theoretical values of $\widehat{\theta\theta}$ at $r=a$.

Angle in degrees.	$\widehat{\theta\theta}$ in fringe order.			
	$a=6$ mm.	$a=5$ mm.	$a=4$ mm.	$a=2{\cdot}5$ mm.
0	0	0	0	0
10	−2·10	−1·76	−1·40	−0·88
20	−3·36	−2·81	−2·24	−1·40
30	−3·22	−2·69	−2·15	−1·34
40	−1·42	−1·19	−0·95	−0·59
45	0	0	0	0
50	1·77	1·47	1·18	0·74
60	5·54	4·63	3·70	2·32
70	9·30	7·75	6·21	3·88
80	12·00	10·00	8·02	5·00
90	12·90	10·77	8·62	5·88

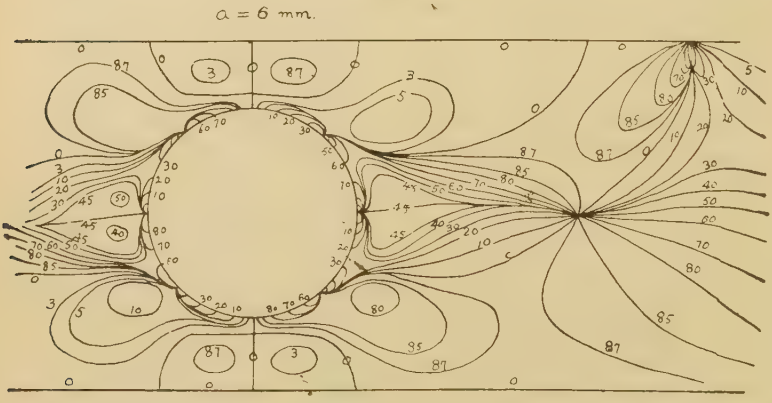
and theoretical values should agree better in the case of a smaller hole than in that of a larger. But, as is clear from fig. 2, the results are contrary to our expectation. It seems that some imperfections in our loading apparatus may be responsible for it. As is shown in fig. 1, the two load points K K′ are connected rigidly to each other; consequently there might have been generated some tangential force (tension) along K K′ when the specimens were deformed, and such force, if any, should act as subtractive against the fibre stress, which was compression in this case. Any unfavourable effect will be shown more conspicuously at a portion

where the fringe order is not high, and the author is of the opinion that the distinct deviation of the experimental values from the theoretical curve in cases of the specimens *d* and *c* is due to this effect.

The locations of the isoclinic lines are at times very ambiguous, as the lines are apt to appear in wide bands, especially with a thick specimen and also when the fringes of higher order are revealed. The author tried with three kinds of specimens—phenolite, xylonite, and glass, all of them being prepared in quite the same dimensions except the thickness.

It was ascertained that the locations of black lines are also quite the same for the three materials, being independent of

Fig. 3.



the thickness, and, moreover, that they were manifested most sharply by the glass specimen. Accordingly, in our cases the diagrams of isoclinic lines were studied by the aid of glass specimens, and one of them (specimen *f*) is shown in fig. 3. The orthogonal trajectories showing the directions of principal stresses were worked out, and are shown in fig. 4.

At the section $x=0$, each of the two stress components is calculated out graphically from the experimental data of $p-q$ by means of the purely optical method for the symmetrical case, using the formula*

$$p = p_0 - \Delta\phi \int_0^A \frac{p-q}{\Delta y}\, ds_1.$$

* L. N. G. Filon and E. G. Coker, Brit. Assoc. Rep. 1914, p. 201; 1922, p. 350.

In the one which is shown in fig. 5 and Table II., the experimental values of $\widehat{xx}$ and $\widehat{yy}$ for the specimen f show

Fig. 4.

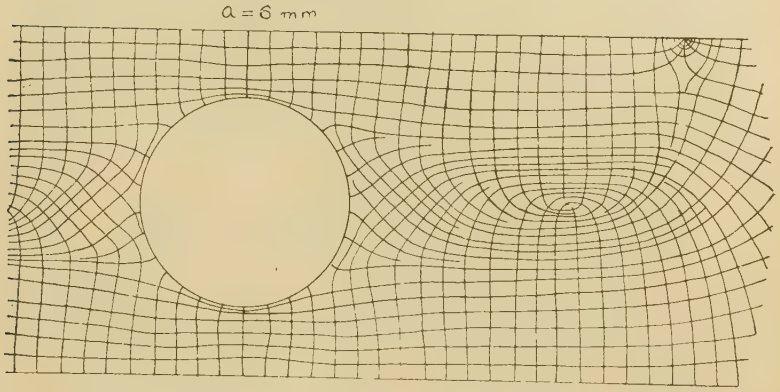
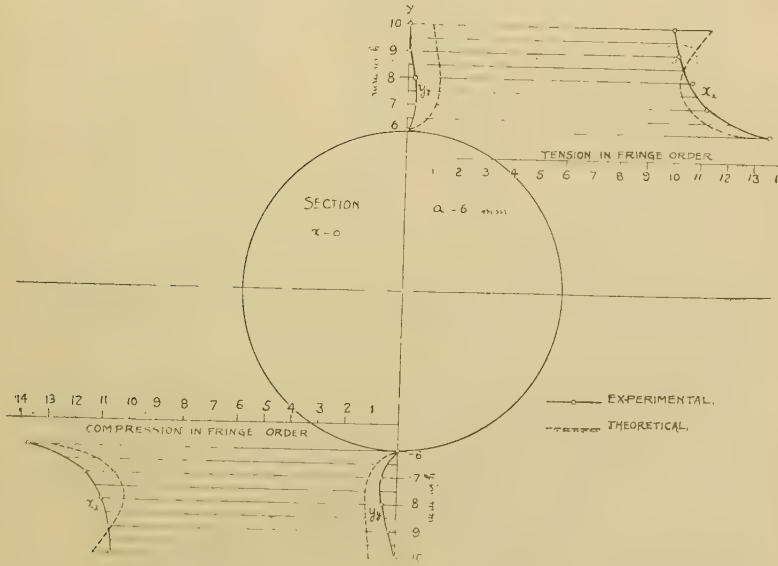


Fig. 5.



considerable deviation from the theoretical, but in this case the deviations are quite plausible, inasmuch as the effect of

the straight boundaries, which had not been taken into consideration in the solution, may act more or less, especially for the specimens of larger holes. On the other hand, however, it is a delicate problem to measure the distances of two adjacent isoclinic lines, Δy, as they are, at least in this section, very critical and not very distinct, so that a slight ambiguity might be the source of considerable errors. Hence the author dares not to set forth, with a firm confidence, the respective values of $\widehat{xx}$ and $\widehat{yy}$, though he

Table II.

Stresses of $\widehat{xx}$ and $\widehat{yy}$ at the section $x=0$ mm.

$a=6$ mm.

y in mm.	Experimental. $\widehat{xx}$ in f. o.	Experimental. $\widehat{yy}$ in f. o.	Theoretical. $\widehat{xx}$ in f. o.	Theoretical. $\widehat{yy}$ in f. o.
10	9·90	0	11·25	0·89
9	10·05	0	10·52	1·06
8	10·60	0·25	10·12	1·19
7	11·30	0·30	10 52	1·08
6	13·50	0	12·90	0
− 6	−13·70	0	−12·90	0
− 7	−11·60	−0·60	−10·52	−1·08
− 8	−11·00	−0·55	−10·12	−1·19
− 9	−10·65	−0·30	−10·52	−1·06
−10	−10·60	0	−11·25	−0·89

could not find any more reliable device to separate each stress component out of the values of $p-q$. At any rate, it is clear in this case that $\widehat{yy}$ is very small compared with $\widehat{xx}$, and it seems to be less important to work out accurately the values of $\widehat{yy}$ except for academic interest.

The other purely optical method preferred by Prof. L. N. G. Filon * is based upon the following formulas :

$$\widehat{xx}-\widehat{xx}_0=-\int_{x_0}^{x}\frac{\partial\widehat{xy}}{\partial y}\,dx,$$

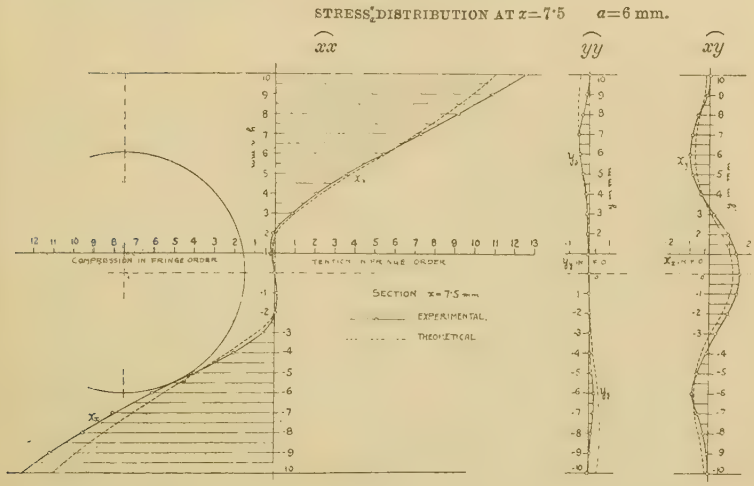
$$\widehat{yy}-\widehat{yy}_0=-\int_{y_0}^{y}\frac{\partial\widehat{xy}}{\partial x}\,dy.$$

* *Loc. cit.*

With the specimen *f*, $a=6$ mm., the stress distribution at the section $x=7{\cdot}5$ mm. was studied by this method, δx being taken as 2 mm. ; two more sections, $x=8{\cdot}5$ mm. and $x=6{\cdot}5$ mm., were studied to work out the values of $\frac{\delta \widehat{xy}}{\delta x}$.

In this case every state was very favourable for the graphical integration, and without any remarkable ambiguities the stress components were worked out as they are shown in fig. 6 and Table III. The theoretical values

Fig. 6.



are also represented by broken lines in the figure, all of which show rather fair agreement with the experimental values.

Summary.

(1) A beam of uniform rectangular section having a circular hole of radius *a* on its centre line was subjected to a uniform bending moment, and its stress distribution was studied by photo-elasticity by means of the photographic method.

(2) From one of the results the form of the stress function was assumed, and a theoretical solution was worked out under the assumption of an infinitely extended plane.

(3) At several sections and boundaries the experimental values were worked out and compared with the theoretical

ones. The author is of the opinion that his solution serves satisfactorily enough for practical purposes, and also that the photographic method is rather preferable, being as free from errors as one can hope in such an experiment.

TABLE III.

Stresses $\widehat{xy}$, $\widehat{yy}$, and $\widehat{xx}$ at the section $x=7{\cdot}5$ mm.

$a=6$ mm.

y in mm.	Experimental values.			Theoretical values.		
	$\widehat{xy}$ in f. o.	$\widehat{yy}$ in f. o.	$\widehat{xx}$ in f. o.	$\widehat{xy}$ in f. o.	$\widehat{yy}$ in f. o.	$\widehat{xx}$ in f. o.
10	0	0	12·50	−0·20	−0·45	10·92
9	−0·04	−0·13	10·86			
8	−0·58	−0·36	9·17	−0·49	−0·58	8·50
7	−0·90	−0·52	7·27			
6	−1·01	−0·48	5·38	−0·72	−0·48	5·52
5	−0·85	−0·31	3·53			
4	−0·40	−0·15	2·06	−0·39	−0·21	2·31
3	−0·07	−0·10	0·89			
2	0·97	−0·09	− 0·09	0·65	−0·09	0·04
1	1·37	−0·05	− 0·14	...	...	− 0·09
0	1·56	0	0	1·19	0	0
− 1	1·37	0·01	0·08	...	...	0·09
− 2	0·93	0·01	0·02	0·65	0·09	− 0·04
− 3	0·33	0·05	− 0·58			
− 4	−0·17	0·04	− 2·01	−0·39	0·21	− 2·31
− 5	−0·58	0·12	− 3·94			
− 6	−0·83	0·28	− 6·05	−0·72	0·48	− 5·52
− 7	−0·55	0·25	− 8·02			
− 8	−0·24	0·10	− 9·48	−0·49	0·58	− 8·50
− 9	−0·02	−0·03	−11·13			
−10	0	−0·05	−13·15	−0·20	0·45	−10·92

In conclusion, the author wishes to express his hearty thanks to Viscount Prof. M. Ōkōchi and to Prof. M. Masima, who gave the author valuable suggestions throughout the course of the experiments.

XXII. *The Critical Frequency in an Ionized Medium. Steady Magnetic Force Present.* *By* T. L. Eckersley, *M.A.** (Marconi's Wireless Telegraph Co., Ltd., Research Department.)

§ 1.

In a recent analysis of the transmission of electromagnetic waves of frequency ν through an ionized medium the well-known critical frequency given by

$$\nu_0^2 = \frac{Ne^2c^2}{\pi m}\dagger$$

(N = no. of electrons per c.c.; e = charge of electron; m = mass of electron) was examined.

It was shown that as ν, the frequency of the waves, approached ν_0, the group velocity $\rightarrow 0$, and that physically this slowing-down of the group velocity was occasioned by the momentum given to the ions. The momentum of the wave travelling, say, in the x direction was transferred to the x momentum of the ions, and at the critical frequency the demand for momentum by the ions is equal to the supply from the wave, and this is therefore brought to a standstill. If a magnetic field is present the critical frequency is altered, and it is the purpose of the present note to show that the altered motion of the ions results in an altered critical frequency, but that the momentum balance is the same as in the previous case, and that the waves are brought to a standstill for the same reason, *i. e.* the transference of the wave momentum to the ions.

For the purpose of this analysis we require the motion of the ions in the field of the electric wave.

As in the previous case, it will not do to neglect action of the magnetic force in the wave, for it is this alone which provides the necessary forward momentum to the ions. The movement of the ions, if this be neglected, is wholly in the plane perpendicular to the ray and the balance of momentum cannot be maintained.

The simplest case to consider is that in which the magnetic force is parallel to the ray. In such a case the wave splits

* Communicated by the Author.

† T. L. Eckersley, "Transmission of Electric Waves through an Ionized Medium," Phil. Mag. iv. (July 1927).

up into two circularly-polarized components, each of which travels with its appropriate group and phase velocity. Each component can be dealt with separately.

Suppose, therefore, that the wave is circularly polarized in the right-hand direction, we shall put

$$Y = Z_0 \cos pt, \qquad X = 0,$$

$$Z = Z_0 \sin pt.$$

The magnetic forces β and γ are

$$\beta = H_0 \sin pt, \qquad \text{(H and Z the same dimensionally.)}$$

$$\gamma = -H_0 \cos pt.$$

The equations of motion for the ions are then

$$m\frac{\partial v_x}{\partial t} = \frac{eH_0}{c} v_z \sin pt + \frac{eH_0}{c} v_y \cos pt$$

$$= \frac{eH_0}{c}\chi_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 1)$$

where χ_1 is an abbreviation for

$$v_z \sin pt + v_y \cos pt.$$

Let R be the steady magnetic field directed along x.

Similarly we have

$$m\frac{dv_y}{dt} = eZ_0 \cos pt - \frac{eRv_z}{c} - \frac{eH_0}{c} v_x \cos pt, \quad . \quad (1, 2)$$

$$m\frac{dv_z}{dt} = eZ_0 \sin pt + \frac{eRv_y}{c} - \frac{eH_0}{c} v_x \sin pt. \quad . \quad (1, 3)$$

Let

$$\frac{eR}{mc} = p_1 = 2\pi\nu_1,$$

where ν_1 is the resonant frequency of the electrons in the magnetic field.

By multiplying (1, 2) by $\sin pt$ and (1, 3) by $\cos pt$ and subtracting and denoting $(v_y \sin pt - v_z \cos pt)$ by χ_2, we get the equation

$$\frac{d\chi_2}{dt} = +(p - p_1)\chi_1. \quad . \quad . \quad . \quad . \quad (1, 4)$$

Similarly, by multiplying (1, 2) by $\cos pt$ and (1, 3) by $\sin pt$ and adding, we get the equation

$$\frac{d\chi_1}{dt} = \frac{eZ_0}{m} + (p_1 - p)\chi_2 - \frac{eH_0}{mc} v_x; \quad . \quad . \quad (1, 5)$$

the three equations are therefore

$$\left.\begin{aligned} \frac{dv_x}{dt} &= \frac{eH_0}{mc}\chi_1, \\ \frac{d\chi_2}{dt} &= (p - p_1)\chi_1, \\ \frac{d\chi_1}{dt} &= \frac{eZ_0}{m} - (p - p_1)\chi_2 - \frac{eH_0}{mc} v_x. \end{aligned}\right\} \quad . \quad . \quad (1, 6)$$

Eliminating v_x and χ_2, we get the following vibrational differential equation for χ_1 :

$$\frac{d^2\chi_1}{dt^2} = (p - p_1)^2\chi_1 - \frac{e^2H_0^2}{m^2c^2}\chi_1. \quad . \quad . \quad . \quad (1, 7)$$

Initial Conditions.

Inverting the equations for χ_1 and χ_2, we get

$$\left.\begin{aligned} v_y &= \chi_1 \cos pt + \chi_2 \sin pt, \\ v_z &= \chi_1 \sin pt - \chi_2 \cos pt, \end{aligned}\right\} \quad . \quad . \quad . \quad (1, 8)$$

so that

$$\left.\begin{aligned} (v_y)_{t=0} &= (\chi_1)_{t=0}, \\ (v_z)_{t=0} &= (\chi_2)_{t=0}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1, 9)$$

We will suppose that the electron starts from rest at the time $t=0$, so that

$$\chi_1 = 0, \quad t = 0,$$
$$\chi_2 = 0, \quad t = 0.$$

The appropriate solution for χ_1 is then

$$\chi_1 = \mathrm{P} \sin \bar{p}t, \quad \text{where } \bar{p}^2 = (p - p_1)^2 + \frac{e^2H^2}{m^2c^2}; \quad (1, 10)$$

now

$$\frac{\partial v_x}{\partial t} = \frac{eH_0}{mc}\chi_1 = \frac{eH_0}{mc}\mathrm{P} \sin \bar{p}t$$

and

$$v_x = \frac{eH_0}{mc}\frac{\mathrm{P}}{\bar{p}}(1 - \cos \bar{p}t), \quad . \quad . \quad . \quad . \quad (1, 11)$$

since $v_x=0$ when $t=0$. Similarly

$$\chi_2 = (p-p_1)\frac{\mathrm{P}}{\bar{p}}(1-\cos\bar{p}t)\,; \quad . \quad . \quad (1,11)'$$

substituting in (1, 5), we get the relation

$$eZ_0 = \frac{(p-p_1)^2}{\bar{p}}\mathrm{P} + \left(\frac{e\mathrm{H}_0}{mc}\right)^2\mathrm{P}, \quad . \quad . \quad (1,12)$$

which determines

$$\mathrm{P} = \frac{\bar{p}eZ_0/m}{(p-p_1)^2+\dfrac{e^2\mathrm{H}_0{}^2}{m^2c^2}} = \frac{eZ_0/m}{\left\{(p-p_1)^2+\dfrac{e^2\mathrm{H}_0{}^2}{m^2c^2}\right\}^{1/2}}.$$

Now the mean forward velocity of the electrons

$$\bar{v} = \frac{e\mathrm{H}_0}{mc}\frac{\mathrm{P}}{\bar{p}},$$

i. e.

$$\bar{v} = \frac{e^2\mathrm{H}_0Z_0}{m^2c}\frac{1}{\left\{(p-p_1)^2+\dfrac{e^2\mathrm{H}_0{}^2}{m^2c^2}\right\}}, \quad . \quad . \quad (1,13)$$

which gives the mean forward momentum supplied to the ions.

This is

$$\mathrm{N}m\bar{v}_x = \frac{\mathrm{N}e^2\mathrm{H}_0Z_0}{mc}\frac{1}{\left\{(p-p_1)^2+\dfrac{e^2\mathrm{H}_0{}^2}{m^2c^2}\right\}}$$

$$= \frac{\pi\mathrm{H}_0Z_0}{c^3}\frac{\nu_0{}^2}{\left\{(p-p_1)^2+\dfrac{e^2\mathrm{H}_0{}^2}{m^2c^2}\right\}}, \quad (1,14)$$

where N is the number of electrons or ions of mass m and charge e per c.c.

It will be observed that to a first approximation $\left(\text{neglecting } \dfrac{e^2\mathrm{H}_0{}^2}{m^2c^2} \text{ compared with } (p-p_1)^2\right)$ the ionic momentum produced is a definite fraction of the wave momentum supplied, *i. e.* H_0Z_0, a fraction which only depends on the critical frequency ν_0, the resonant frequency ν_1, and the actual frequency ν.

A similar analysis may be made for the case where the

circular polarization is opposite. This is equivalent to reversing R, and we get

$$\mathrm{N}m\bar{v}_x = \frac{\pi \mathrm{H}_0 \mathrm{Z}_0}{c^3} \frac{{v_0}^2}{\left\{ (p+p_1)^2 + \frac{e^2 \mathrm{H}_0^2}{m^2 c^2} \right\}} \cdot \quad . \quad (1, 15)$$

It might be thought that the result depended on rather arbitrary initial conditions.

In fact, by specifying $t=0$ as the time of arrival of the waves at the electrons, we have fixed $\mathrm{Y}=0$, $\mathrm{Z}=\mathrm{Z}_0$ at this time.

If at $t=0$, Y and Z had been arbitrarily phased, say

$$\mathrm{Y}_z = \mathrm{Z}_0 \cos(pt+\phi), \qquad \mathrm{Z} = \mathrm{Z}_0 \sin(pt+\phi),$$

so that

$$\mathrm{Y} = \mathrm{Z}_0 \cos\phi \cos pt - \mathrm{Z}_0 \sin\phi \sin pt,$$

$$\mathrm{Z} = \mathrm{Z}_0 \sin\phi \cos pt + \mathrm{Z}_0 \cos\phi \sin pt.$$

Then referred to axes rotated about x, for an angle ϕ,

$$\left.\begin{aligned} \mathrm{Y}_1 &= \quad \mathrm{Y}\cos\phi + \mathrm{Z}\sin\phi = \mathrm{Z}_0 \cos pt, \\ \mathrm{Z}_1 &= -\mathrm{Y}\sin\phi + \mathrm{Z}\cos\phi = \mathrm{Z}_0 \sin pt, \end{aligned}\right\}$$

as before, and, as the choice of axes is arbitrary, we may take Y_1 and Z_1 instead of Y and Z, and get exactly the same results as previously.

§ 2. *Balance of Momentum.*

As in the previous paper, we find that the momentum supplied per unit area is $\frac{\mathrm{Z}_0 \mathrm{H}_0}{2\pi c}$. The rate of increase of momentum at the head of the wave is $\frac{\mathrm{Z}_0 \mathrm{H}_0}{2\pi c} \frac{\mathrm{U}}{c}$, where U is the group velocity, since $\frac{\mathrm{Z}_0 \mathrm{H}_0}{2\pi c^2}$ is the momentum density and the head of the wave travels with the velocity U.

The head of the wave is only a sharply-defined surface moving forward with the velocity U when U does not differ largely from c; also in this case the departure from linearity of the equations is small. For these reasons the momentum conditions can only be simply calculated when U and c do not differ much. The condition where $\mathrm{U} \to 0$ near the critical frequency cannot be considered without going into a great deal more detail in calculating the exact distribution of the wave-energy and the consequent ionic movements.

We therefore restrict ourselves to the case where U does not differ largely from c, or where $\frac{\nu_0^2}{\nu(\nu \pm \nu_1)}$ is small (these two conditions being equivalent, as we shall see).

In the first place we must calculate the group velocity.

As is well known, the phase velocity in these conditions is

$$v = \frac{c}{\sqrt{1 - \frac{\nu_0^2}{\nu(\nu \pm \nu_1)}}}, \quad . \quad . \quad . \quad . \quad (2,1)$$

where the symbols have their previous meanings, and where the + or − sign is taken according as the wave is circularly polarized in a right-hand or left-hand direction. The group velocity is given in general, as is well known, by the relation

$$U = \frac{d\nu}{d\frac{\nu}{v}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,2)$$

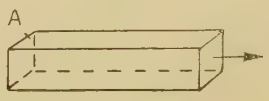
Using the above value of v, we get for the group velocity

$$U = c\sqrt{1 - \frac{\nu_0^2}{\nu(\nu \pm \nu_1)}} \Big/ \left\{1 \mp \frac{1}{2}\frac{\nu_0^2 \nu_1}{\nu(\nu \pm \nu_1)^2}\right\} . \quad (2,3)$$

It is to be noted that $Uv \neq c^2$ in this case.

Consider a column of radiation 1 sq. cm. in section extended along the direction of transmission.

Fig. 1.



Head end of wave moving with velocity U.

The rate of supply of æthereal momentum at A is $\frac{Z_0 H_0}{2\pi c}$; at the head end of the wave the rate of gain of æthereal momentum is $\frac{Z_0 H_0}{2\pi c}\frac{U}{c}$, and the deficit in æthereal momentum is

$$\frac{Z_0 H_0}{2\pi c}(1 \quad U/c) ; \quad . \quad . \quad . \quad . \quad . \quad (2,4)$$

the gain in ionic momentum is

$$\mathrm{N}mv_x = \frac{\mathrm{Z}_0\mathrm{H}_0}{4\pi c} \frac{\nu_0^2}{\left\{ (\nu \pm \nu_1)^2 + \frac{e^2\mathrm{H}_0^2}{4\pi^2 m^2 c^2} \right\}} . \quad (1, 14,) \ \& \ (1, 15)$$

Now by (2, 3), when

$$\frac{\nu_0^2}{\nu(\nu \pm \nu_1)} \text{ and } \frac{\nu_0^2\nu_1}{\nu(\nu \pm \nu_1)^2}$$

are small, $(1 - \mathrm{U}/c)$ is

$$(a) \quad 1 - \{(1 - \tfrac{1}{2}\nu_0^2/\nu(\nu - \nu_1))/(1 + \tfrac{1}{2}\nu_0^2\nu_1/\nu(\nu - \nu_1)^2)\}, \quad (2, 5)$$

$$(b) \quad 1 - \{(1 - \tfrac{1}{2}\nu_0^2/\nu(\nu + \nu_1))/(1 - \tfrac{1}{2}\nu_0^2\nu_1/\nu(\nu + \nu_1)^2)\}; \quad (2, 6)$$

i. e.

$$\left.\begin{array}{ll} (a) & \frac{1}{2}\frac{\nu_0^2}{(\nu - \nu_1)^2}, \\ (b) & \frac{1}{2}\frac{\nu_0^2}{(\nu + \nu_1)^2}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2, 7)$$

so that the deficit in æthereal momentum is to the first approximation,

$$\frac{\mathrm{Z}_0\mathrm{H}_0}{4\pi c} \frac{\nu_0^2}{(\nu \pm \nu_1)^2},$$

and, except for the negligible term $\frac{e^2\mathrm{H}_0^2}{4\pi^2 m^2 c^2}$, is identical with the gain in the ionic momentum.

Extending these results to the case where $\mathrm{U} \to 0$, we may conclude that the group velocity of the wave $\to 0$ on account of the demand for momentum of the ions which tend to rob the wave of all its momentum (in the neighbourhood of $\nu =$ initial frequency) so that the wave can travel no further. This investigation is important from the point of view of the transmission of waves in an over-dense medium.

In the absence of a magnetic field the medium ceases to be transparent if N, the ionic density, is so great that $\nu_0 > \nu$. The medium may be said to be over dense in such a case.

If in such a medium a sufficiently strong magnetic field is introduced that $\frac{\nu_0}{\nu(\nu + \nu_1)}$ is small, the medium becomes transparent (for a suitably circularly-polarized ray). The physical reason for this transparency is now clear, for whereas when no magnetic field is present the amount of

x momentum produced by the waves, *i. e.* $\frac{Z_0 H_0}{4\pi c}\frac{\nu_1^2}{\nu^2}$, is greater than can be supplied by the wave, the ionic movement is so altered in the presence of the magnetic field that the forward ionic momentum is reduced in the ratio $\frac{\nu^2}{(\nu+\nu_1)^2}$. By increasing ν_1 the demand for x momentum by the ions can be made so small that the momentum of the wave is not appreciably reduced.

It can therefore travel through the over-dense ionized medium when a sufficiently strong magnetic field is introduced.

These results have only been deduced for the particular case where the ray direction is actually along the direction of the steady magnetic field.

Transmission at any Angle θ with the Ray.

When the ray direction makes an angle θ with the direction of R the analysis is more complex, but it is possible to show that, but for the exceptional case where $\nu_1 \doteq \nu_0 \doteq \nu$, the above is not unique.

Thus examining the phase-velocity equation, we find that the phase velocity, group velocity X, Y, and Z all vary continuously in the neighbourhood $\theta=0$, and therefore it follows that the transparency is maintained over a finite region near $\theta=0$, the physical reason for this transparence being that stated above, *i. e.* the modification of the ionic motions by the magnetic field in such a way that the demand for momentum of the ions is decreased below what can be supplied by the wave.

It seems to me to follow quite definitely that the application of a sufficiently intense magnetic field to an over-dense medium will make the latter transparent for a finite range of angles in accordance with the dictates of the phase-velocity equations.

This result has an important application in the case of the transmission of waves over the earth's surface, for in the case of waves >100 m. we know that the layer is over-dense, *i.e.*, $N \geq 10^5$ electrons per c.c. Although longer waves generated on the earth's surface would be wholly confined to the region between the earth and Heaviside layer in the absence of a magnetic field, sufficiently long waves can penetrate the layer when the earth's magnetic field is taken into account.

XXIII. *The Electronic Theory of Valency.*—Part VII. *The Etch-Figures of Sylvine. By* Professor T. M. Lowry, *F.R.S., and* M. A. Vernon, *B.A.**

[Plates III. & IV.]

UNTIL Bragg published the results of the X-ray analysis of rock-salt it was generally believed that the alkali halides crystallized in hemihedral forms of the cubic system. This conclusion was based on the observation that the etch-figures were not symmetrical, but were rotated in a definite direction with respect to the planes of symmetry of the crystal. The structure deduced from X-ray analysis, however, consists of two interpenetrating face-centred cubic lattices which have a complete holohedral symmetry. In the same way crystallographers had concluded from observations on twinned crystals of diamond that the crystals had a hemihedral symmetry, whereas X-ray analysis indicated that the lattice was holohedral. These observations were of importance in that they suggested the possible existence of a fine structure, which was not disclosed by X-ray analysis of the crystal-lattice, although it might be detected by the traditional methods used by crystallographers in order to determine the symmetry of crystals.

In order to resolve this apparent contradiction Herzfeld and Hettich, in 1926 †, made experiments on the etch-figures of sylvine. As a result of their experiments they suggested that the lack of symmetry of the etch-figures was not due to a lack of symmetry in the crystal structure, but to the presence of optically active impurities in the etching solution, which formed crystallographically unsymmetrical adsorption compounds on the surface of the salt. Thus, when specially purified water was used, their etch-figures were all perfectly symmetrical; but, when traces of optically-active impurities were introduced, a large number of unsymmetrical etch-figures were observed. These conclusions were questioned by J. P. Valetone ‡, but were confirmed by the more detailed report of Hettich §. The matter seemed, however, to be sufficiently important to

* Communicated by the Faraday Society.
† *Z. Physik*, p. 38 (1926).
‡ *Ibid.* xxxix. p. 60.
§ *Z. Krist.* lxiv. p. 265 (1926).

call for further independent evidence, and the present work was undertaken for this purpose.

Experiments were first made with natural crystals of sylvine. The material was by no means perfect, and the results obtained were rather irregular. As recommended by Hettich, the water was purified by distillation from alkaline permanganate, and the glass vessels were cleansed with chromic acid and rinsed with purified water. The crystals were etched with a saturated solution of pure potassium chloride to which a few drops of purified water were added. It was observed that solutions which had been saturated by boiling with the solid salt were much less satisfactory for etching than those made by shaking with the salt in the cold, but we cannot give any explanation of this difference of behaviour. The crystal was placed on a watch-glass on the stage of a microscope, covered with the solution, and the etching process watched. Since the etch-figures of sylvine are usually convex and not concave, an oblique illumination was used in order to show up the pyramidal prominences. In one isolated instance, however, when natural sylvine was etched in presence of amyl alcohol one or two pyramidal pits were observed.

In order to get the best results it is necessary to use a fresh and perfect cleavage, and this was very difficult to obtain with the imperfect natural material. Some trouble was also caused by the tendency of the solution to deposit crystals on the surface, perhaps as a result of the large number of nuclei of crystallization provided by the irregularities on the cleavage-face. It was found, however, that nearly all the pyramids were symmetrically placed with respect to the cleavage-edges. Occasionally a few of the pyramids appeared to be twisted, but there was considerable variation in the angle of rotation of these unsymmetrical pyramids, even on the same crystal-face. A possible explanation of this anomaly is provided by the very significant fact that in some cases pyramids were seen to detach themselves bodily from the crystal, and to float a little distance before finally coming to rest in an unsymmetrical position. This was only observed when natural crystals were used, and was probably due to the presence of cleavage-cracks below these particular pyramids.

Traces of a series of optically active alcohols and esters

were next introduced into the etching solution. No appreciable alteration was observed, except perhaps a slight increase in the number of etch-figures in a given area. Specimen counts were made of the numbers of symmetrical and rotated pyramids in the field of view, and some of the results are given below. These results show clearly that the etch-figures of natural sylvine are in general perfectly symmetrical.

Impurity.	Number of Pyramids.		
	Symmetrical.	Rotated Left.	Rotated Right.
None	40	0	1
,,	30	0	1
,,	50	1	2
l-Amyl alcohol (natural)	45	3	3
d-γ-Nonyl formate	60	0	1
d-n-Butyl-vinyl-carbinol	60	0	3
l-n-Butyl-vinyl-carbinol	60	2	1
l-n-Butyl-vinyl-carbinol formate	50	0	1
l-Ethyl *3*-phenethyl propionate	50	0	1

Experiments were then made with artificial crystals. For this purpose potassium chloride was purified by repeated recrystallization from water, dried and pulverized. It was placed in a clean platinum dish and fused in a muffle-furnace. When the fused salt had been allowed to cool the top layer was a clear transparent mass, whilst below it was an agglomeration of tiny crystals. Pieces of the top layer were detached, using a carefully cleaned knife-blade and tweezers. Small cubical fragments were thus obtained, the surfaces of which were formed by exceedingly good and well-marked cleavages. The difficulties arising in the previous case from crystallization on the surface and from the floating away of pyramidal hillocks were now entirely absent, and very good etch-figures were obtained, but perhaps less rapidly than with the natural crystals. Although a large number of experiments were performed, both with and without impurities in the etching solution, *not a single unsymmetrical pyramid was observed.*

Photographs, taken with a Zeiss camera attachment to the microscope, illustrate the results obtained. In

Pl. III., where *l*-asparagine was added to the etching solution, a large number of well-developed pyramids are seen, which are all oriented alike. The cleavage-edge does not come into the field of view, which was in the centre of the crystal face, but the cleavage was found to be parallel to the square edges of the pyramids. In the experiments illustrated in Pl. IV. the impurities were (*a*) and (*b*) none, (*c*) and (*d*) amyl alcohol, (*e*) ethyl tartrate, and (*f*) *l*-asparagine. In some cases the edge of the crystal fragment is seen, instead of a cleavage-edge; but the true direction of the cleavage can be deduced from cleavage-cracks in the field of view, and this direction has been marked by a horizontal line on each photograph.

The evidence now adduced is of a different character from that of Hertzfeld and Hettich, since in our experiments we were unable to develop unsymmetrical etch-figures, whereas they were only able to get rid of them by taking special precautions. In spite of this difference, however, the two series of experiments lead to the same conclusion, namely, that crystals of sylvine do not contain any structural element which leads to the regular development of unsymmetrical etch-figures, and that when these are observed they are of a fortuitous character, which does not justify the conclusion that the crystal-lattice of the salt is unsymmetrical.

In this connexion it is desirable to emphasize the fortuitous way in which etch-figures are necessarily developed. Thus, in his recent '*Leçons de Cristallographie*' Friedel, who has made an exceptionally thorough study of the subject, finds it necessary to issue a warning in reference to the pitfalls which are associated with this method of investigating crystal-symmetry. In particular, whilst it has generally been held that the occasional appearance of unsymmetrical pitting must outweigh any number of observations in which symmetrical pits are seen, Professor Friedel insists that symmetrical pitting can only be expected on surfaces that are accurately oriented, and that, even in the case of calcite, the true cleavage-planes are interrupted by small conchoidal sections on which the etching is necessarily unsymmetrical.

The matter can, however, be carried one stage further, since on a perfectly homogeneous face there would be no irregularities from which either pitting or the development of hillocks could start, and the etch-figures would only be

of molecular dimensions. Thus, even in their purest synthetic crystals Herzfeld and Hettich found that pyramidal hillocks were only formed at points where there were minute specks of ferric oxide to check the initial action of the etch-liquid on the soluble salt. This fact would not invalidate the evidence derived from a lower order of symmetry in the etch-figures, if this could be shown to be a constant property of the crystal, but our own experiments agree with those of Herzfeld and Hettich in showing that this is not the case. We therefore conclude that the higher symmetry established by X-ray analysis cannot be challenged on the basis of data provided by the study of etch-figures.

In the case of rock-salt, then, there is at present no trustworthy crystallographic evidence which can be used to prove the presence of real molecules in the crystal. The properties of the highly-conducting liquid are also in harmony with the conception of an ionic aggregate, differing only from the solid in being mobile and disordered, instead of rigid and orderly. On the other hand, measurements of conductivity show that liquid water contains only one pair of ions for every 550 million molecules, and even this small proportion diminishes as the temperature falls. There is therefore no analogous justification for supposing that ice is "a packing of charged ions O^{--}, H^{+}" *, since this implies that the molecules, which make up practically the whole content of liquid water, disappear completely in the act of crystallization. Physico-chemical evidence, on the contrary, indicates that the formation of ice is accompanied by a process of polymerization, which is already in progress in the liquid state, and provides an obvious explanation of the very exceptional phenomenon of maximum density. The conclusion that ice is formed from water by polymerization rather than by ionization is also in harmony with the crystallographic evidence, since the open lattice, which is disclosed by X-ray analysis, is much more compatible with a network of bonds than with an aggregation of close-packed ions, such as is postulated in a crystal of fluor spar. Since, however, an increase in the valency of oxygen is always accompanied by the development of a positive charge, as in the formation of oxonium ions, $\overset{\prime}{O}H_2 \rightarrow \overset{+}{O}H_3 \rightarrow \overset{++}{O}H_4$, it appears probable

* Gibbs, Proc. R. S., A. cxiii. p. 361 (1926).

that, if the atoms of oxygen and hydrogen in ice are electrically charged, the quadrivalent oxygen must carry a positive charge and the bivalent hydrogen a negative charge, in place of the charges of opposite sign which would normally be carried by free ions of these elements *.

The crystal-structure of quartz presents an analogous problem. Gibbs † has assigned to it an ionic structure, which can be justified by the fact that silicon (unlike hydrogen) does in fact possess some of the properties of a metal. The contrast between this ionic structure and the molecular structure of carbon dioxide serves, moreover, to explain the wide differences between the physical properties of the two oxides, as illustrated by the rise from a sublimation-point at $-78°$ to a boiling-point at $+2600°$ C., which occurs on passing from carbon dioxide to silicon dioxide. Nevertheless it is almost as difficult to believe that the bonds of the silicate ion disappear completely in silicic anhydride as to believe that the bonds of the water-molecule disappear completely in ice. This difficulty can, however, again be avoided by making use of Thomson's conception of intramolecular ionization ‡, which was discussed somewhat fully in the first paper of the present series §. In the case of carbon dioxide the formation of a covalent molecule with non-polar double bonds between carbon and oxygen is comparatively easy ; but G. N. Lewis has pointed out that "the ability to form multiple bonds is almost entirely, if not entirely, confined to elements of the first period of eight, and especially to carbon, nitrogen, and oxygen" ‖. This conclusion has been confirmed by the recent experiments of Professor F. S. Kipping on "The Carbon-Silicon Binding" ¶, in reference to which he says that "Fresh evidence is continually being obtained . . . that an ethylenic binding between carbon and silicon is either impossible or can

* Sugden (J. Chem. Soc. 1929, p. 316) has recently suggested a scheme in which the atoms of ice are all neutral. This depends on reducing the bonds between the atoms to *one* shared electron, instead of the *two* electrons postulated by Lewis's theory of covalency. The scheme therefore involves a novel point of view for which no sufficient experimental evidence is yet available.

† Gibbs, Proc. R. S., A. cxiii. p. 361 (1926).

‡ Phil. Mag. [6] xxvii. p. 757 (1914).

§ Phil. Mag. xlv. p. 1105 (1923).

‖ 'Valence,' p. 94 (1923).

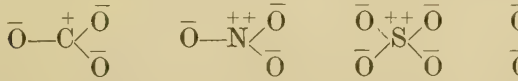
¶ J. Chem. Soc. p. 104 (1927).

only be produced under exceptional conditions. Those reactions which lead to the formation of an olefine seem to be quite inapplicable to the production of the group $>C:Si<$." There is therefore a direct experimental justification for adopting Lewis's suggestion that "if silicon is incapable of sharing a double bond with oxygen . . . we should be obliged to represent silicon dioxide as

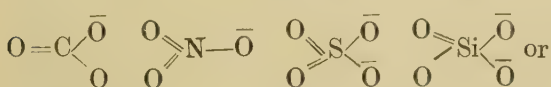
$$—O—\underset{|}{\overset{|}{Si}}—O—,$$

and the uncompleted bonds could only be completed by union with other molecules." From this point of view the molecule of silica, with its four free bonds, resembles the atom of carbon, and should give rise to a network of the same general type as that of diamond or graphite. It would be interesting to know whether this network of bonds is compatible with the crystal structures assigned to silica in its various forms, since it appears *à priori* that such a network would be far more likely than a mere aggregate of ions to give rise to the complex polymorphism which is actually observed.

In conclusion, attention may be redirected to the formulæ assigned in the first paper of this series to the carbonate, nitrate, sulphate, and silicate ions



These formulæ, in which charged atoms are united by bonds, have the merit of satisfying the conditions of crystal-symmetry better than the traditional chemical formulæ



although the latter may perhaps represent more accurately the structure of the ions in solution when removed from the influence of the symmetrically-placed kations of the crystal. At the same time they avoid the difficulties, both chemical and physical, which arise from the attempt to represent all these anions as metallic kations, $\overset{++++}{C}$ $\overset{+++++}{N}$ $\overset{++++++}{S}$ and $\overset{++++}{Si}$, surrounded by a surplus of oxide ions $\overset{--}{O}$. In particular, the formulæ for the nitrate

and carbonate ions, whilst conforming strictly to the crystal-symmetry of calcite and of sodium nitrate, enable us to avoid the illogical conclusion that the real bonds of carbonic anhydride and of the oxides of nitrogen disappear completely in the carbonates and nitrates.

Summary.

On etching natural crystals of sylvine with a strong solution of potassium chloride unsymmetrical pyramids were sometimes formed by the flotation of detached fragments, but on artificial crystals the etch-figures were always symmetrical, even in presence of optically-active impurities. There are therefore no trustworthy crystallographic data which can be used as evidence of a lower order of symmetry than that deduced from X-ray analysis, or as a proof of the existence of bonds between individual atoms of potassium and of chlorine. The ionic structure assigned to sylvine is also in harmony with the high conductivity of the fused salt.

On the other hand, objection is taken to the view that crystals of ice and quartz, and the anions of oxygenated acids in crystals such as calcite and barytes, are mere aggregates of oxide ions $\dot{\dot{O}}^{--}$ with kations such as $\dot{H}^{+}$, $\dot{C}^{++++}$, $\dot{Si}^{++++}$, $\dot{N}^{+++++}$, and $\dot{S}^{++++++}$. Thus the formation of ice is more likely to be a process of polymerization in which additional bonds are formed than a process of ionization in which all the bonds between the atoms are destroyed; a network of single bonds between quadrivalent oxygen and bivalent hydrogen is therefore postulated. In the same way it is suggested that the oxygen atoms in the carbonate, nitrate, sulphate, and silicate ions are linked to the central atom by single bonds, and therefore carry only single negative charges. A network of single bonds is also postulated in quartz, but in this case the atoms of quadrivalent silicon and bivalent oxygen would be electrically neutral.

One of us (M. A. V.) is indebted to the London County Council for a scholarship held during the period in which this work was carried out.

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XXIV. *Some Problems in the Conduction of Heat.* By George Green, *D.Sc., Lecturer in Physics in the Applied Physics Department of the University of Glasgow* *.

In two previous communications † relating to the subject of heat conduction illustrations have been given of the method of solving problems by means of wave-trains. In these earlier papers the main purpose in view was to develop the fundamental solutions which generally form the basis from which we derive or build up the solution of particular problems, the application of the method to important or difficult problems being left to a later opportunity. The method is useful in all cases of plane, spherical, or cylindrical heat distributions, and it has the advantage that the mathematical solutions are arrived at by direct application of the familiar physical ideas associated with the transmission of a wave-train across a boundary or with its reflexion from the boundary. In addition, the normal coordinates, which are in general required for the solution of problems in which the conduction of heat in a limited portion of matter is under consideration, and are obtained usually by a separate investigation, are derived in the natural course of application of this method. The particular form which they take in each special case becomes evident when the boundary conditions are applied to each constituent wave-train of a given frequency, as we shall see in the special cases treated below. In the present paper the main purpose in view is to demonstrate the suitability of the method and to show its power in the treatment of problems of considerable interest and difficulty. The problems chosen for this purpose have in most cases, though not in all, been solved already by other methods, so that the relative values of the two methods of solution of heat-conduction problems can be readily determined.

The equation of heat conduction in a uniform medium may be written in the form

$$\frac{\partial v}{\partial t} = \kappa \nabla^2(v), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\kappa = \mathrm{K}/c\rho$; v, K. c, ρ being temperature, conductivity, specific heat, and density respectively. This equation, in one form or other, together with equations representing

* Communicated by the Author.

† Phil. Mag. iii. Suppl. pp. 784–800 (April 1927): pp. 701–720 (April 1928).

special boundary conditions, has to be satisfied in all cases to be considered. As the first problem to be examined we take that of determining the temperature in a finite rod of length a, having an initial temperature distribution given by $f(x)$, and under a condition at the boundaries $x=0$ and $x=a$, which may be that radiation takes place there to a medium at zero temperature, or that the temperature there is maintained at zero, or any similar condition. To solve this problem we start from the solution representing a plane periodic heat source qe^{ik} situated at plane $x=x_1$ within the medium, given in the usual notation by

$$v = \frac{q}{2K\sqrt{\frac{ik}{\kappa}}} e^{ikt-(x-x_1)\sqrt{\frac{ik}{\kappa}}} \quad \text{for } x > x_1, \quad (2)$$

$$v = \frac{q}{2K\sqrt{\frac{ik}{\kappa}}} e^{ikt-(x_1-x)\sqrt{\frac{ik}{\kappa}}} \quad \text{for } x_1 > x. \quad (2')$$

The first represents a positive wave-train supplying $\frac{1}{2}qe^{ikt}$ heat units per unit area at $x=x_1$ in the positive direction of x from x_1, and the second represents a negative wave-train supplying an equal amount in the negative direction from x_1. Follow now the positive train alone. At $x=a$ some boundary condition (such as $v=0$) has to be fulfilled, and as the above train alone violates this condition, it must give rise to a reflected wave-train, so that, instead of a single positive train (2), we require

$$v = \rho e^{ikt-(x-x_1)i\lambda} + \mathrm{A}\rho e^{ikt-(2a-x_1-x)i\lambda}, \quad (3)$$

in which we have written ρ instead of $\frac{q}{2K\sqrt{\frac{ik}{\kappa}}}$, and $i\lambda$ instead of $\sqrt{\frac{ik}{\kappa}}$. The additional term now appearing represents the negative wave-train reflected from boundary $x=a$, A being an operator or boundary coefficient whose value remains to be determined by the special boundary condition chosen. The reflected wave-train is also to be regarded as the continuation within the medium of the original wave-train as modified and redirected by the boundary. The negative wave-train introduced above in turn fails to satisfy the required boundary condition at $x=0$, and in turn gives rise to a reflected wave-train in the positive

direction of x. To include the effect of this second reflexion, the original positive train is now to be accompanied by two wave-trains, which are its continuations within the medium, so that from (3) we pass to

$$v = \rho e^{ikt-(x-x_1)i\lambda} + A\rho e^{ikt-(2a-x_1-x)i\lambda} + A^2\rho e^{ikt-(2a-x_1+x)i\lambda}. \quad (4)$$

The phase of each wave-train agrees with the phase of the corresponding reflected wave-train at the boundary at which reflexion takes place; thus in the above cases we have

$$x - x_1 = 2a - x_1 - x \quad \text{at } x = a,$$

and

$$2a - x_1 - x = 2a - x_1 + x \quad \text{at } x = 0.$$

At each boundary a continuing wave-train arises, and an additional term is required to represent its effect within the medium. The complete wave-system arising from the original positive wave-train by successive reflexions at boundaries $x=a$ and $x=0$ respectively is represented by

$$v_1 = \rho e^{ikt-(x-x_1)i\lambda}\{1 + A^2e^{-2ai\lambda} + A^4e^{-4ai\lambda} + A^6e^{-6ai\lambda} + \text{etc.}\}$$
$$+ A\rho e^{ikt-(2a-x_1-x)i\lambda}\{1 + A^2e^{-2ai\lambda} + A^4e^{-4ai\lambda} + A^6e^{-6ai\lambda} + \text{etc}\}. \quad (5)$$

In a similar manner we find that the complete wave-system arising from the original negative wave-train by successive reflexions at boundaries $x=0$ and $x=a$ respectively is represented by

$$v_2 = \{\rho e^{ikt-(x_1-x)i\lambda} + A\rho e^{ikt-(x_1+x)i\lambda}\}\left\{\frac{1}{1-A^2e^{-2ai\lambda}}\right\}, \quad (6)$$

in which we have inserted the sum of the geometric series appearing in (5). The sum of the two expressions contained in (5) and (6) represents the temperature at any point x at instant t due to a periodic source qe^{ikt} situated at x_1, the boundaries $x=0$ and $x=a$ being subject to some definite condition which determines the value of the coefficient A. The temperature at time t at any point x due to an instantaneous surface source q initially at x_1, subject to the same boundary conditions, is obtained directly from the above results by an integration indicated by

$$v = \frac{1}{\pi}\int_0^\infty dk(v_1 + v_2). \quad (7)$$

When we make use of the transformation

$$\sqrt{\frac{ik}{\kappa}} = i\lambda$$

throughout, and combine corresponding terms in v_1 and v_2, it will readily be found that (7) becomes

$$v = \frac{q\kappa}{K\pi}\int_0^\infty d\lambda\, e^{-\kappa\lambda^2 t}\{\cos(x-x_1)\lambda + Ae^{-ai\lambda}\cos(a-x-x_1)\lambda\} \times \frac{1}{1-A^2e^{-2ai\lambda}}. \quad . \quad . \quad (8)$$

This is essentially the solution we require in integral form, including all possible values of A. The evaluation of the integral appearing in (8) can most directly be carried out by contour integration, the path of integration with respect to λ being along axes inclined at 45° to the real axis of λ, and along two circular arcs connecting these axes at infinity.

We have now to assign definite values to A in accordance with the particular conditions of reflexion of each wave-train at the boundaries of the medium. If the wave-trains are reflected under the condition that at the boundaries the temperature is maintained at zero ($v=0$ at $x=0$ and $x=a$), then $A=-1$: and the positive roots of $1-e^{-2ai\lambda}=0$ are at the values of λ given by $a\lambda=0$, π, 2π, 3π, etc. For all such values of λ we have also

$$\frac{d}{d\lambda}\{1-e^{-2ai\lambda}\} = 2ai,$$

and accordingly, by means of the theorem of residues, the evaluation of (8) is given by

$$v = \frac{q\kappa}{Ka}\sum_1^\infty e^{-\kappa\lambda^2 t}\{\cos(x-x_1)\lambda - \cos(x+x_1)\lambda\}$$

where $a\lambda=0, \pi, 2\pi$, etc.

$$= \frac{2q\kappa}{Ka}\sum_1^\infty e^{-\kappa\left(\frac{n\pi}{a}\right)^2 t}\sin\frac{n\pi}{a}x\sin\frac{n\pi}{a}x_1, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the summation extending to all values of n, in agreement with the well-known result. If, on the other hand, the wave-trains are reflected at the boundaries under the condition that radiation takes place there into a medium at zero temperature

$$\left\{-K\frac{\partial v}{\partial x} = hv\right\},$$

then

$$A = \frac{iK\lambda - h}{iK\lambda + h},$$

and the roots of the equation

$$1-A^2e^{-2ai\lambda} = 0$$

are identical with the roots of the equation

$$\tan a\lambda = \frac{2Kh\lambda}{K^2\lambda^2 - h^2}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Moreover, corresponding to each value of λ given by the roots of (10), we have

$$\frac{d}{d\lambda}\{e^{ai\lambda} - A\} = -\frac{i\{a(K^2\lambda^2 + h^2) + 2Kh\}}{(iK\lambda + h)^2}, \quad . \quad (11)$$

and we can find readily from these results, by applying the residue theorem, that the evaluation of (8) is now given by

$$v = \frac{q\kappa}{K}\Sigma e^{-\kappa\lambda^2 t}\{\cos(x - x_1)\lambda + \cos(a - x - x_1)\lambda\} \times \frac{K^2\lambda^2 + h^2}{a(K^2\lambda^2 + h^2) + 2Kh}, \quad . \quad . \quad (12)$$

the summation extending to all the roots of (10) above. Since we can put

$$\cos a\lambda = \frac{K^2\lambda^2 - h^2}{K^2\lambda^2 + h^2} \quad \text{and} \quad \sin a\lambda = \frac{2Kh\lambda}{K^2\lambda^2 + h^2},$$

it can easily be verified that the above result is equivalent to

$$v = \frac{2q\kappa}{K}\sum_1^\infty e^{-\kappa\lambda^2 t} \times \frac{(K\lambda\cos x\lambda + h\sin x\lambda)(K\lambda\cos x_1\lambda + h\sin x_1\lambda)}{a(K^2\lambda^2 + h^2) + 2Kh}, \quad (13)$$

which is also in agreement with a known result. If in the above results we put $q = 1$, these solutions, (9) and (13), then are Green's Functions corresponding each to a special boundary condition. From the above treatment it is evident that the method employed is perfectly general, being applicable to any arbitrary boundary conditions. To return to the problem stated at the outset, we have merely to replace q in (9) and (13) by $f(x_1)\,dx_1$ and integrate with respect to x_1 from $x_1 = 0$ to $x_1 = a$, and so obtain the temperature at x at instant t due to an initial arbitrary heat distribution in the material.

Consider next the problem of determining the temperature at any point within a sphere of radius a, having an initial temperature distribution represented by $v = f(r)$, and subject to a boundary condition at surface $r = a$ which can be specified later. The investigation proceeds along exactly

the same lines as in the problem treated above. That is, we start from the solution representing a periodic heat source qe^{ikt} per unit area, situated at any surface $r=r_1$ within the sphere, namely,

$$v = \frac{qr_1}{2\mathrm{K}r\sqrt{\frac{ik}{\kappa}}} e^{ikt-(r-r_1)\sqrt{\frac{ik}{\kappa}}} = \rho e^{ikt-(r-r_1)i\lambda} \quad . \quad r>r_1, \quad (14)$$

$$v = \rho e^{ikt-(r_1-r)i\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad r<r_1. \quad (14')$$

The first part of the solution represents a positive wave-train supplying $\frac{1}{2}qe^{ikt}$ heat units per unit of area at r_1, and the second part represents a negative wave-train supplying an equal quantity of heat in the opposite direction from r_1. Consider the positive train alone. After one reflexion at the boundary $r=a$, the effect of the original train and its continuation is given by

$$\rho e^{ikt-(r-r_1)i\lambda} + \mathrm{A}\rho e^{ikt-(2a-r_1-r)i\lambda}, \quad . \quad . \quad . \quad (15)$$

the additional train being the negative wave-train reflected from the boundary $r=a$, and agreeing in phase with the original wave-train at $r=a$. As we have pointed out in an earlier paper, each negative wave-train must, after passing through the origin, continue as a positive train; hence from (15) we pass to

$$\rho e^{ikt-(r-r_1)i\lambda} + \mathrm{A}\rho e^{ikt-(2a-r_1-r)i\lambda} - \mathrm{A}\rho e^{ikt-(2a-r_1+r)i\lambda}, \quad (16)$$

in which the new wave-train appearing is a positive train. The process is repeated by each successive positive train, so that we can write down at once the wave-system within the sphere arising from the original positive wave-train in the form

$$\rho e^{ikt+r_1i\lambda}\{e^{-ri\lambda} + 2i\sin\lambda r.\,\mathrm{A}e^{-2ai\lambda}(1-\mathrm{A}e^{-2ai\lambda}+\mathrm{A}^2e^{-4ai\lambda}-\text{etc.})\}.$$
$$. \quad . \quad . \quad (17)$$

The corresponding expression for the complete wave-system arising from the original negative wave-train is given by

$$v_2 = 2i\sin\lambda r\,\rho e^{ikt-r_1i\lambda}\{1-\mathrm{A}e^{-2ai\lambda}+\mathrm{A}^2e^{-4ai\lambda}-\text{etc.}\}. \quad (18)$$

In these, (17) and (18), it is evident that the trains are arranged in pairs according to powers of A, a positive and a negative together, in order to fulfil the conditions required at the origin that the temperature there is finite. The original positive train alone, represented by the first term of

(17), violates this condition, and we have accordingly to bring this term into conformity with the condition by introducing the corresponding negative train. This means that instead of (17), which is incomplete, we write the complete wave-system arising from the original positive wave-train in the form

$$v_1 = -2i \sin \lambda r \rho e^{ikt + r_1 i\lambda} \cdot \frac{1}{1 + Ae^{-2ai\lambda}} \cdot \quad . \quad . \quad (19)$$

Just as we found in the previous problem, the sum of the two expressions contained in (18) and (19) represents the temperature at r at instant t due to a periodic source qe^{ikt} per unit area situated at r_1: and from this, by an integration with respect to k, indicated by (7) above, we find the temperature at r at instant t due to an instantaneous surface source q per unit area situated at $r = r_1$ in the form

$$v = \frac{4q\kappa r_1}{Kr\pi} \int_0^\infty d\lambda \, e^{-\kappa\lambda^2 t} \frac{\sin \lambda r \sin \lambda r_1}{1 + Ae^{-2ai\lambda}} \cdot \quad . \quad . \quad (20)$$

Taking $A = -1$, as before, corresponding to the condition that the surface of the sphere $r = a$ is maintained at zero, we find that the poles of the integrand are again given by $a\lambda = 0, \pi, 2\pi, \ldots n\pi$, etc. The evaluation of the integral required is again effected by contour integration along the path, consisting of the two axes inclined to the real axis of λ at angles of 45°, connected by a circular arc at infinity. The result obtained by means of the Residue theorem is

$$v = \frac{2q\kappa r_1}{Kr} \sum_1^\infty e^{-\kappa\lambda^2 t} \sin \lambda r \sin \lambda r_1, \quad . \quad . \quad . \quad (21)$$

where $\lambda = n\pi/a$, and the summation extends to all values of n.

Taking

$$A = \left\{ i\lambda - \left(\frac{1}{a} + \frac{h}{K} \right) \right\} \Big/ \left\{ i\lambda - \left(\frac{1}{a} - \frac{h}{K} \right) \right\},$$

corresponding to the condition that radiation takes place at surface $r = a$ into a medium at zero temperature, we find that the poles of the integrand are at values of λ given by the the roots of the equation

$$\tan 2a\lambda = - \frac{2\lambda \left(\frac{h}{K} - \frac{1}{a} \right)}{\left\{ \left(\frac{h}{K} - \frac{1}{a} \right)^2 - \lambda^2 \right\}} \quad \text{or} \quad \tan a\lambda = - \frac{\lambda}{\left(\frac{h}{K} - \frac{1}{a} \right)}.$$

$$. \quad . \quad . \quad (22)$$

At each value of λ given by (22) we find that

$$\frac{d}{d\lambda}\{e^{2ai\lambda}+\mathrm{A}\}$$
$$=i\left[2a\left\{\lambda^2+\left(\frac{h}{\mathrm{K}}-\frac{1}{a}\right)^2\right\}+2\left(\frac{h}{\mathrm{K}}-\frac{1}{a}\right)\right]\Big/\left\{i\lambda+\left(\frac{h}{\mathrm{K}}-\frac{1}{a}\right)\right\}^2, \quad . \quad . \quad . \quad (23)$$

and accordingly, by means of the theorem of Residues, we find that the evaluation of (20) is given by

$$v=\frac{2q\kappa r_1}{\mathrm{K}ra}\sum_1^\infty e^{-\kappa\lambda^2 t}\sin\lambda r\sin\lambda r_1 \times\frac{\mathrm{K}^2a^2\lambda^2+(ah-\mathrm{K})^2}{\{\mathrm{K}^2a^2\lambda^2+(ah-\mathrm{K})^2+\mathrm{K}a(ah-\mathrm{K})\}}, \quad (24)$$

the summation extending to all the roots of (22) above. It will be seen that the results arrived at in (21) and (24) are in agreement with known results. To obtain the solution corresponding to an initial heat distribution throughout the sphere, we have merely to replace q by $f(r_1)\,dr_1$ and integrate with respect to r_1 from 0 to a.

As a further application of the same method we take the case of a sphere of one material enclosed within another material bounded also by a concentric spherical surface, the whole having an initial temperature distribution $f(r)$, and, as before, subject to any specified condition at the outer boundary of the outer material. We take K_1,κ_1 to refer to the inner material bounded by the surface $r=a$, and K_2,κ_2 to refer to the outer material bounded by the surfaces $r=a$ and $r=b$. The complete solution aimed at consists of four parts. Starting with a periodic heat source qe^{ikt} situated at r_1 within medium 1, we require the temperature at each instant (a) at any point r within medium 1, and (b) at any point r within medium 2. Similarly, starting with a periodic heat source $q'e^{ikt}$ situated at r_1 within medium 2, we require the temperature at each instant (a) at any point r within medium 2, and (b) at any point r within medium 1. Taking first the case of a source at r_1 in medium 1, we build up our solution from two initial wave-trains corresponding exactly to (14) and (14′) above. If we introduce the necessary modifications in our notation indicated by

$$\left.\begin{aligned}\rho_1&=\frac{r_1}{2\mathrm{K}_1r}\sqrt{\frac{ik}{\kappa_1}};\qquad \rho_2=\frac{r_1}{2\mathrm{K}_2r}\sqrt{\frac{ik}{\kappa_2}};\\ i\lambda&=\sqrt{\frac{ik}{\kappa_1}};\qquad \mu i\lambda=\sqrt{\frac{ik}{\kappa_2}},\end{aligned}\right\} \quad . \quad (25)$$

the expression representing the wave-system which arises from the two fundamental wave-trains within medium 1 by successive reflexions at the boundary $r=a$ is identical with the sum of (18) and (19) above, already included in (20), namely,

$$v_1 = 4q\rho_1 e^{ikt} \sin\lambda r \sin\lambda r_1 \frac{1}{1+Ae^{-2ai\lambda}}. \quad . \quad . \quad (26)$$

This gives the primary effect of the source at r_1 at any point r within medium 1 itself.

First Effects Transmitted into Medium 2.

This effect is produced, as we have seen, by the group of positive trains

$$q\rho_1 e^{ikt}\{e^{-(r-r_1)i\lambda} - e^{-(r+r_1)i\lambda}\}\frac{1}{1+Ae^{-2ai\lambda}}, \quad . \quad . \quad (27)$$

accompanied by the corresponding group of negative trains reflected from boundary $r=a$. Each of these positive wave-trains transmits a continuing wave-train into medium 2. For example, let

$$v_1 = q\rho_1 e^{ikt-(r-r_1)i\lambda} + Aq\rho_1 e^{ikt-(2a-r_1-r)i\lambda} \quad . \quad . \quad (28)$$

represent a positive train and the reflected train at boundary $r=a$, and let

$$v_2 = A'q\rho_2 e^{ikt-(a-r_1)i\lambda-\mu(r-a)i\lambda} \quad . \quad . \quad . \quad (29)$$

represent the corresponding continuing wave-train transmitted into medium 2, then the two boundary coefficients A and A′ are to be determined by the conditions

$$v_1 = v_2 \quad \text{and} \quad K_1\frac{\partial v_1}{\partial r} = K_2\frac{\partial v_2}{\partial r} \quad \text{at} \quad r=a. \quad . \quad (30)$$

Their values will be given later. Meantime we see from the above that the first group of wave-trains transmitted from medium 1 into medium 2 is represented by

$$\left.\begin{array}{l} A'q\rho_2 e^{ikt-\mu(r-a)i\lambda}\{e^{-(a-r_1)i\lambda} - e^{-(a+r_1)i\lambda}\}\dfrac{1}{1+Ae^{-2ai\lambda}} \\ \text{or} \qquad\qquad \alpha' e^{-\mu(r-a)i\lambda}. \end{array}\right\} \quad (31)$$

This group of wave-trains, being positive, first undergoes reflexion at $r=b$, and then is partly reflected and partly transmitted into medium 1 at boundary $r=a$. Through successive reflexions at boundaries b and a the wave-system

in medium 2 is built up. Taking the boundary coefficient for reflexion at b as B, and that for reflexion at a as C′, we can represent the original wave-group in medium 2 and its continuations by

$$\left.\begin{array}{lll} \alpha' e^{-\mu(r-a)i\lambda} & \text{followed by} & \mathrm{B}\,\alpha' e^{-\mu(2b-a-r)i\lambda}, \\ \mathrm{BC}'\,\alpha' e^{-\mu(2b-3a+r)i\lambda} & ,, \quad ,, & \mathrm{B^2C}'\,\alpha' e^{-\mu(4b-3a-r)i\lambda}, \\ \mathrm{B^2C}'^2\,\alpha' e^{-\mu(4b-5a+r)i\lambda} & ,, \quad ,, & \mathrm{B^3C}'^2\,\alpha' e^{-\mu(6b-5a-r)i\lambda}, \end{array}\right\} \quad (32)$$

and so on, the successive positive waves being given in the left-hand column and the successive negative waves in the right-hand column. The summations of the positive and negative waves are given by

$$\alpha' e^{-\mu(r-a)i\lambda}/(1-\mathrm{BC}'e^{-2\mu(b-a)i\lambda}) \quad . \quad . \quad . \quad . \quad (33)$$

and

$$\mathrm{B}\alpha' e^{-\mu(2b-a-r)i\lambda}/(1-\mathrm{BC}'e^{-2\mu(b-a)i\lambda}) \quad . \quad . \quad (33')$$

respectively. By combining these we obtain the complete effect at r in medium 2 of the first transmission into medium 2 from medium 1 in the form

$$\mathrm{V}_1 = 2iq\rho_2 e^{ikt-\{a+\mu(b-a)\}i\lambda} \sin\lambda r_1 \{e^{\mu(b-r)i\lambda} + \mathrm{B}e^{-\mu(b-r)i\lambda}\}.\,\mathrm{A}'\mathrm{S}_1\mathrm{S}_2,$$

where . . . (34)

$$\mathrm{S}_1 = 1/\{1+\mathrm{A}e^{-2ai\lambda}\} \quad \text{and} \quad \mathrm{S}_2 = 1/\{1-\mathrm{BC}'e^{-2\mu(b-a)i\lambda}\}. \quad (35)$$

First Effects Retransmitted into Medium 1.

Consider now the group of negative wave-trains (33′) forming part of the wave-system (34) in medium 2. Each negative wave-train, on reaching boundary $r=a$, transmits a negative train into medium 1. For example, with C as boundary coefficient for transmission from medium 2 into medium 1, the negative train $\rho_2 e^{-\mu(2b-a-r)i\lambda}$ in medium 2 continues into medium 1 as the train represented by

$$\mathrm{C}\rho_1 e^{-\mu(2b-2a)i\lambda-(a-r)i\lambda}.$$

We accordingly represent the first negative wave-system returning into medium 1 by

$$q\rho_1 e^{ikt-2\{a+\mu(b-a)\}i\lambda+ri\lambda}\{e^{r_1 i\lambda} - e^{-r_1 i\lambda}\}.\,\mathrm{A}'\mathrm{BC}\,\mathrm{S}_1\mathrm{S}_2. \quad (36)$$

These trains, being negative, continue their course through the origin, and thereafter become positive trains represented by

$$-q\rho_1 e^{ikt-2\{a+\mu(b-a)\}i\lambda-ri\lambda}\{e^{r_1 i\lambda} - e^{-r_1 i\lambda}\}.\,\mathrm{A}'\mathrm{BC}\,\mathrm{S}_1\mathrm{S}_2, \quad (37)$$

and as they continue their course by multiple reflexions at boundary $r=a$, in exactly the manner worked out in the case of the original positive train (14), they build up within medium 1 the system

$$v_2 = -4q\rho_1 e^{ikt} \sin \lambda r \sin \lambda r_1 . e^{-2\{a+\mu(b-a)\}i\lambda} . \mathrm{A'BC\,S_1{}^2S_2}, \quad (38)$$

which represents the first secondary effect of the source at r_1 at any point r within medium 1 itself, the primary effect being given by v_1 in (26) above. We can evidently rewrite (38) in the form

$$v_2 = -\alpha v_1, \quad \text{where } \alpha = e^{-2\{a+\mu(b-a)\}i\lambda} . \mathrm{A'BC\,S_1S_2}. \quad (39)$$

The process described above for the trains represented by v_1 is now repeated by the trains represented by v_2. The positive trains included in v_2, given in (38), again transmit wave-trains into medium 2, where they in turn build up a wave-system whose total effect is given by $\mathrm{V_2}$, where $\mathrm{V_2} = -\alpha \mathrm{V_1}$, $\mathrm{V_1}$ being given by (34). The corresponding return wave-system contributes the second secondary effect, or the third actual effect, of the source at r_1 to the total effect at any point r in medium 1. Denoting this by v_3, we have

$$\left.\begin{aligned} v_3 &= -\alpha v_2 \;\; = \alpha^2 v_1, \\ \text{and similarly} \qquad \mathrm{V_3} &= -\alpha \mathrm{V_2} = \alpha^2 \mathrm{V_1}, \\ &\text{and so on.} \end{aligned}\right\} \quad . \; . \; . \quad (40)$$

The coefficient α can be used, like the other coefficients, simply as an operator. The summation of all the terms v_1, v_2, v_3, etc. (and also that of the terms $\mathrm{V_1}$, $\mathrm{V_2}$, $\mathrm{V_3}$, etc.) is now a simple matter. If we put

$$\mathrm{S} = 1 - \alpha + \alpha^2 - \alpha^3 + \text{ etc.},$$

we can now write down the total effect at r in medium 1 due to a source qe^{ikt} situated at r_1 in medium 1 in the form

$$v = 4q\rho_1 e^{ikt} \sin \lambda r \sin \lambda r_1 \, \mathrm{S_1 S}, \qquad \mathrm{S} = 1/(1+\alpha), \quad (41)$$

and the total effect at r in medium 2, due to the same source at r_1 in medium 1, in the form

$$v = 2iq\rho_2 e^{ikt - \{a+\mu(b-a)\}i\lambda} \sin \lambda r_1 \{ e^{\mu(b-r)i\lambda} + \mathrm{B} e^{-\mu(b-r)i\lambda} \} \mathrm{A'S_1 S_2 S}. \quad . \; . \; . \quad (42)$$

These results contain the solutions required in the case of a periodic source qe^{ikt} situated at r_1 in medium 1. The corresponding results representing the effects of a periodic source $q'e^{ikt}$ situated at r_1 in medium 2 can be obtained by exactly the same process from the initial trains

$$q'\rho_2 e^{-\mu(r-r_1)i\lambda} \quad \text{and} \quad q'\rho_2 e^{ikt-\mu(r_1-r)i\lambda}. \quad . \quad . \quad (43)$$

These wave-trains are to be regarded as continuing after reflexions at boundaries $r=b$ and $r=a$ respectively. The original trains and their first three or four continuations in medium 2 are indicated in order in the vertical columns :—

$$\left.\begin{array}{ll} \left\{\begin{array}{l} q'\rho_2 e^{ikt-\mu(r-r_1)i\lambda}, \\ \mathrm{B}q'\rho_2 e^{ikt-\mu(2b-r_1-r)i\lambda}, \end{array}\right. & \left\{\begin{array}{l} q'\rho_2 e^{ikt-\mu(r_1-r)i\lambda}, \\ \mathrm{C}'q'\rho_2 e^{ikt-\mu r_1+(r-2a)i\lambda}, \end{array}\right. \\ \left\{\begin{array}{l} \mathrm{BC}'q'\rho_2 e^{ikt-\mu(2b-2a-r_1+r)i\lambda}, \\ \mathrm{B}^2\mathrm{C}'q'\rho_2 e^{ikt-\mu(4b-2a-r_1-r)i\lambda}, \end{array}\right. & \left\{\begin{array}{l} \mathrm{BC}'q'\rho_2 e^{ikt-\mu(2b-2a+r_1-r)i\lambda}, \\ \mathrm{BC}'^2q'\rho_2 e^{ikt-\mu(2b-4a+r_1+r)i\lambda}, \end{array}\right. \end{array}\right\}$$

. . . (44)

and so on, B being boundary coefficient determined by the conditions to be fulfilled at $r=b$, and C′ being boundary coefficient for wave-trains reflected into medium 2 at $r=a$. Each positive wave-train in the above system, after reaching boundary $r=b$, continues as a negative train ; hence the above wave-trains occur in pairs, each pair consisting of a positive followed by a negative wave-train, which together satisfy the boundary condition at $r=b$. The original negative wave-train $q'\rho_2 e^{ikt-\mu(r_1-r)i\lambda}$ alone violates this condition. As before, in the case of the first term of (17), to bring the anomalous term into conformity with the condition required at $r=b$, the corresponding positive term must therefore be introduced at the top of the right-hand column above, namely,

$$\frac{q'\rho_2}{\mathrm{B}} e^{ikt+\mu(2b-r_1-r)i\lambda}.$$

The summation of the above terms then gives the complete primary effect produced in medium 2 by a periodic source situated at r_1 in medium 2, the result obtained being given by

$$q'\rho_2 e^{ikt}\left\{e^{-\mu(b-r_1)i\lambda}+(1/\mathrm{B})e^{\mu(b-r_1)i\lambda}\right\}\left\{e^{\mu(b-r)i\lambda}+\mathrm{B}e^{-\mu(b-r)i\lambda}\right\}\mathrm{S}_2.$$

. . . (45)

Each negative wave-train contained in the above wave-system transmits a wave-train into medium 1, and secondary

effects are produced in medium 2 by wave-trains retransmitted from medium 1. The process has been explained above, and we need only write down here the final result, representing the complete effect at r in medium 2, due to a periodic source $q'e^{ikt}$ situated at r_1 in medium 2, in the form

$$v=q'\rho_2 e^{ikt}\{e^{-\mu(b-r_1)i\lambda}+(1/\mathrm{B})e^{\mu(b-r_1)i\lambda}\}\{e^{\mu(b-r)i\lambda}+\mathrm{B}e^{-\mu(b-r)i\lambda}\}\mathrm{S}_2\mathrm{S}. \quad . \quad . \quad . \quad (46)$$

Similarly the complete effect at r in medium 1, due to a periodic source $q'e^{ikt}$ situated at r_1 in medium 2, may be written in the form

$$v=2iq'\rho_1 e^{ikt-\{a+\mu(b-a)i\lambda}\sin\lambda r\{e^{\mu(b-r_1)i\lambda}+\mathrm{B}e^{-\mu(b-r_1)i\lambda}\}\mathrm{CS}_1\mathrm{S}_2\mathrm{S}, \quad . \quad . \quad . \quad (47)$$

where C is the boundary coefficient for wave-trains transmitted from medium 2 into medium 1 at $r=a$. To determine C and C′, we put

$$v_2=q'\rho_2 e^{ikt-\mu(r_1-r)i\lambda}+\mathrm{C}'q'\rho_2 e^{ikt-(r_1+r-2a)i\lambda} \quad . \quad (48)$$

to represent a negative train and its continuation by reflexion at $r=a$ in medium 2, and

$$v_1=\mathrm{C}q'\rho_1 e^{ikt-\mu(r_1-a)i\lambda-(a-r)i\lambda} \quad . \quad . \quad . \quad (49)$$

to represent its continuation into medium 1, and apply to v_1 and v_2 the conditions expressed in (30) above: the same conditions used to determine A and A′. The expression (47) and the earlier expression (42) are reciprocal, as we shall see from the evaluations of the constants C and A given later. Each expression is converted into the other by the simple interchange of r_1 and r, excepting, of course, where r_1 and r appear in ρ_1 and ρ_2.

Our system of results contained in (41), (42), (46), (47) is now complete so far as periodic sources in medium 1 and medium 2, enclosed within finite boundaries, are concerned. To arrive at the corresponding results for instantaneous sources q_1 and q', we have to evaluate two integrals of the form $\frac{1}{\pi}\int_0^\infty dk\,v$ for any point r in medium 1, and two corresponding integrals for any point r in medium 2. For the effect at r in medium 1 of an instantaneous source q per unit area at surface $r=r_1$ also in medium 1, we have from (41) above,

$$v=\frac{4q\kappa_1 r_1}{\pi \mathrm{K}_1 r}\int_0^\infty d\lambda\sin\lambda r_1\sin\lambda r e^{-\kappa_1\lambda^2 t}\cdot\frac{1-\mathrm{BC}'e^{-2\mu(b-a)i\lambda}}{\mathrm{D}} \quad . \quad (50)$$

For the effect at r in medium 1 of an instantaneous source q' per unit area at surface $r=r_1$ in medium 2, we find from (47)

$$v = \frac{2iq'\kappa_1 r_1}{\pi \mathrm{K}_1 r} \int_0^\infty d\lambda \sin \lambda r e^{-\kappa_1\lambda^2 t - \{a+\mu(b-a)\}i\lambda} \cdot \{e^{\mu(b-r_1)i\lambda} + \mathrm{B}e^{-\mu(b-r_1)i\lambda}\} \cdot \frac{\mathrm{C}}{\mathrm{D}}. \quad (51)$$

For the effect at r in medium 2 of an instantaneous source q' per unit area at surface $r=r_1$ also in medium 2, we find from (46)

$$v = \frac{4q'\kappa_2 r_1 \mu}{\pi \mathrm{K}_2 r} \int_0^\infty d\lambda e^{-\kappa_1\lambda^2 t} \{(1/\mathrm{B})\, e^{\mu(b-r_1)i\lambda} + e^{-\mu(b-r_1)i\lambda}\} \cdot \{e^{\mu(b-r)i\lambda} + \mathrm{B}e^{-\mu(b-r)i\lambda}\} \frac{1+\mathrm{A}e^{-2ai\lambda}}{\mathrm{D}}, \quad (52)$$

and for the effect at r in medium 2 of an instantaneous source q per unit area at surface $r=r_1$ in medium 1, we find from (42)

$$v = \frac{2iq\kappa_2 r_1 \mu}{\pi \mathrm{K}_2 r} \int_0^\infty d\lambda \sin \lambda r_1 e^{-\kappa_1\lambda^2 t - \{a+\mu(b-a)\}i\lambda} \cdot \{e^{\mu(b-r)i\lambda} + \mathrm{B}e^{-\mu(b-r)i\lambda}\} \cdot \frac{\mathrm{A}'}{\mathrm{D}}. \quad (53)$$

In each of the above integrals the value of D is given by

$$\mathrm{D} = (1+\mathrm{A}e^{-2ai\lambda})(1-\mathrm{BC}'e^{-2\mu(b-a)i\lambda}) + \mathrm{A}'\mathrm{BC}e^{-2\{a+\mu(b-a)i\lambda}$$

$$= \frac{1}{\mathrm{S}_1\mathrm{S}_2\mathrm{S}}. \quad (54)$$

The conditions determining the four coefficients A, A′, C, C′, have already been given. The four integrals (50), (51), (52), (53) contain the solutions to a number of different problems according to the values assigned to the boundary coefficient B. When B has the value -1, the solutions apply to the case where an initial heat source exists at r_1, either in medium 1 or medium 2, while the boundary $r=b$ is maintained at zero temperature. When B is given the value

$$\frac{u\lambda + \left(\frac{1}{b} - \frac{h}{\mathrm{K}_2}\right)}{i\mu\lambda - \left(\frac{1}{b} - \frac{h}{\mathrm{K}_2}\right)},$$

the solutions apply to the case where an initial heat source occurs at r_1, while radiation takes place at boundary $r=b$ into a medium kept at zero temperature. When we put $B=0$, and make b infinitely large, the solutions above can be modified to apply to the case of a sphere of radius a surrounded by a medium of different material extending to infinity, but in this case the solutions are more readily obtained by a direct application of the method employed in the present paper, particularly in the case corresponding to (52), where the initial source is in the outer medium. The solutions for the case of an initial source at r_1 in medium 1, the outer medium extending to infinity, are obtained from (50) and (53) above by simply putting $B=0$ in both.

To enable us to carry out the evaluation of the four main integrals constituting the solutions of certain problems, we require now the values of the various boundary coefficients and of the expressions depending upon them contained in the integrals. If for convenience we write

$$\left\{ K_1\sqrt{\kappa_2}+K_2\sqrt{\kappa_1}-\frac{\sqrt{\kappa_2}}{ai\lambda}(K_1-K_2) \right\} = d,$$

we find that

$$\left.\begin{aligned} A &= \frac{\left\{ K_1\sqrt{\kappa_2}-K_2\sqrt{\kappa_1}+\frac{\sqrt{\kappa_2}}{ai\lambda}(K_1-K_2) \right\}}{d}, \\ A' &= \frac{2K_2\sqrt{\kappa_1}}{d}; \\ C &= \frac{2K_1\sqrt{\kappa_2}}{d}, \\ C' &= \frac{\left\{ K_1\sqrt{\kappa_2}-K_2\sqrt{\kappa_1}-\frac{\sqrt{\kappa_2}}{ai\lambda}(K_1-K_2) \right\}}{d}; \end{aligned}\right\} \quad (55)$$

so that

$$A = C-1 \quad \text{and} \quad A' = C'+1,$$

and

$$1+Ae^{-2ai\lambda} = 2e^{-ai\lambda}\left\{ K_1\sqrt{\kappa_2}\cos a\lambda-\frac{\sqrt{\kappa_2}}{a\lambda}(K_1-K_2)\sin a\lambda + iK_2\sqrt{\kappa_1}\sin a\lambda \right\}/d. \quad (56)$$

Other values required depend on the particular value chosen for B, and in what follows we have taken

$$B = \frac{i\mu\lambda + \left(\frac{1}{b} - \frac{h}{K_2}\right)}{i\mu\lambda - \left(\frac{1}{b} - \frac{h}{K_2}\right)} \quad \text{or} \quad \frac{i\mu\lambda + \left(\frac{1}{b} - \frac{h}{K_2}\right)}{d'} \quad . . \quad (57)$$

Corresponding to this value of B we find that

$$D = 4ie^{-\{a+\mu(b-a)\}i\lambda}\left[\mu\lambda f_1(\lambda) - \left(\frac{1}{b} - \frac{h}{K_2}\right) f_2(\lambda)\right] \Big/ dd', \quad (58)$$

where

$$\begin{aligned} f_1(\lambda) = {} & K_1\sqrt{\kappa_2}\cos a\lambda \cos\mu(b-a)\lambda \\ & - K_2\sqrt{\kappa_1}\sin a\lambda \sin\mu(b-a)\lambda \\ & - \frac{\sqrt{\kappa_2}}{a\lambda}(K_1 - K_2)\sin a\lambda\cos\mu(b-a)\lambda, \quad . \quad (59) \end{aligned}$$

$$\begin{aligned} f_2(\lambda) = {} & K_1\sqrt{\kappa_2}\cos a\lambda \sin\mu(b-a)\lambda \\ & + K_2\sqrt{\kappa_1}\sin a\lambda \cos\mu(b-a)\lambda \\ & - \frac{\sqrt{\kappa_2}}{a\lambda}(K_1 - K_2)\sin a\lambda\sin\mu(b-a)\lambda, \quad . \quad (60) \end{aligned}$$

$$\begin{aligned} & 1 - BC'e^{-2\mu(b-a)i\lambda} \\ & = 2e^{-\mu(b-a)i\lambda}\Big[i\mu\lambda\left\{\left(K_1\sqrt{\kappa_2} - \frac{\sqrt{\kappa_2}}{ia\lambda}(K_1 - K_2)\right)\right. \\ & \left. \times\cos\mu(b-a)\lambda + iK_2\sqrt{\kappa_1}\sin\mu(b-a)\lambda\right\} \\ & - \left(\frac{1}{b} - \frac{h}{K_2}\right)\left\{K_2\sqrt{\kappa_1}\cos\mu(b-a)\lambda\right. \\ & \left. + i\left(K_1\sqrt{\kappa_2} - \frac{\sqrt{\kappa_2}}{ai\lambda}(K_1 - K_2)\right)\sin\mu(b-a)\lambda\right\}\Big] \Big/ dd', \\ & \qquad\qquad . \quad . \quad . \quad (61) \end{aligned}$$

$$\begin{aligned} & e^{-\{a+\mu(b-a)\}i\lambda}\left\{e^{\mu(b-r_1)i\lambda} + Be^{-\mu(b-r_1)i\lambda}\right\}/D \\ & = \frac{\left\{\mu\lambda\cos\mu(b-r_1)\lambda - \left(\frac{1}{b} - \frac{h}{K_2}\right)\sin\mu(b-r_1)\lambda\right\} K_1\sqrt{\kappa_2}}{\left\{\mu\lambda f_1(\lambda) - \left(\frac{1}{b} - \frac{h}{K_2}\right) f_2(\lambda)\right\}}. \\ & \qquad\qquad . \quad . \quad . \quad (62) \end{aligned}$$

The evaluation of the integrals with which we are concerned can most directly be effected by contour integration, as in previous problems, along a path consisting of two axes inclined at 45° to the real axis of λ, and the circular arcs joining them at infinity. It is evident, therefore, that the solutions (50) ... (53) then consist of an infinite series of terms, the value of each term being determined by a value of λ which is a root of the equation

$$\mu\lambda f_1(\lambda) - \left(\frac{1}{b} - \frac{h}{K_2}\right) f_2(\lambda) = 0. \quad . \quad . \quad . \quad (63)$$

When we make use of this equation the expressions required can be greatly simplified. Corresponding to each value of λ given by a root of (63), we find that we can write

$$\frac{1 - BC' e^{-2\mu(b-\quad)\lambda}}{D} = \frac{-\mu\lambda K_1 \sqrt{\kappa_2} \cos \mu(b-a)\lambda + \left(\frac{1}{b} - \frac{h}{K_2}\right) \sin \mu(b-a)\lambda}{2i \left\{\mu\lambda f_1(\lambda) - \left(\frac{1}{b} - \frac{h}{K_2}\right) f_2(\lambda)\right\} \sin a\lambda}, \quad . \quad . \quad . \quad (64)$$

$$\frac{1 + Ae^{-2ai\lambda}}{D} = \frac{\{f_1(\lambda) + if_2(\lambda)\} d'}{2i \left\{\mu\lambda f_1(\lambda) - \left(\frac{1}{b} - \frac{h}{K_2}\right) f_2(\lambda)\right\}} \quad . \quad . \quad (65)$$

These results facilitate the evaluation of the integrals above by the residue theorem, for the value of B chosen above. The same results may still be made use of when B is given the value -1, for if in (57) we omit $\mu\lambda$ throughout, the expression reduces to $B = -1$. Similarly the evaluations of the various terms apply to the case where $B = -1$, when terms containing $\mu\lambda$ as a multiplier are omitted. Thus, instead of (63), we have now simply

$$f_2(\lambda) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

as the equation whose roots are required for the evaluation of the integrals (50) ... (53), when B has the value -1.

The two equations (63) and (66) are particular cases of the more general equation $D = 0$, where D is given by (54) above. Another case of interest is that in which $B = 0$ and b is infinitely large, corresponding to a finite sphere enclosed

within an infinite medium. The equation whose roots are required is then

$$(1 + Ae^{-2ai\lambda}) = 0,$$

or its equivalent given in (56). A detailed investigation of the nature of the roots of equation $D=0$ in all its various forms would of course be required in order to determine completely worked-out expressions for all the solutions contained in the integrals given above, but this is beyond the scope and intention of the present paper. We accordingly conclude this demonstration of the value of the method we have employed by completing the evaluations required in the special case in which $B=-1$. These evaluations are contained in

$$v = \sum_1^\infty \frac{2q\kappa_1 r_1}{K_1 r} \frac{\sin\lambda r \sin\lambda r_1 e^{-\kappa_1\lambda^2 t}}{f_2'(\lambda)} \times -K_1\sqrt{\kappa_2}\frac{\sin\mu(b-a)\lambda}{\sin a\lambda}, \quad . \quad . \quad . \quad (67)$$

$$v = \sum_1^\infty -\frac{2q'\kappa_1 r_1}{K_1 r} \frac{\sin\lambda r \sin\mu(b-r_1)\lambda e^{-\kappa_1\lambda^2 t}}{f_2'(\lambda)} K_1\sqrt{\kappa_2}, \quad . \quad . \quad (68)$$

$$v = \sum_1^\infty \frac{2q'\kappa_2\mu r_1}{K_2 r} \frac{\sin\mu(b-r)\lambda \sin\mu(b-r_1)\lambda e^{-\kappa_1\lambda^2 t}}{f_2'(\lambda)} \times -K_2\sqrt{\kappa_1}\frac{\sin a\lambda}{\sin\mu(b-a)\lambda}, \quad (69)$$

$$v = \sum_1^\infty -\frac{2q\kappa_2\mu r_1}{K_2 r} \frac{\sin\lambda r_1 \sin\mu(b-r)\lambda e^{-\kappa_1\lambda^2 t}}{f_2'(\lambda)} . K_2\sqrt{\kappa_1}, \quad . \quad (70)$$

in which the summations extend to all the roots of equation (66) above. These represent respectively, in the order given,

(1) the temperature at r in medium 1 due to an instantaneous source q per unit area at r_1 in medium 1 ;

(2) the temperature at r in medium 1 due to an instantaneous source q' per unit area at r_1 in medium 2 ;

(3) the temperature at r in medium 2 due to an instantaneous source q' per unit area at r_1 in medium 2 ;

(4) the temperature at r in medium 2 due to an instantaneous source q per unit area at r_1 in medium 1.

The solutions are the equivalents of Green's Functions corresponding to a special boundary condition, namely, that the temperature is maintained at value zero at boundary

$r=b$; and they are fundamental solutions from which solutions to any problem of arbitrary initial heat distribution within the two media may be obtained. If we suppose, for example, that initially both media are at a uniform temperature v_0 throughout, we have merely to replace q by $\frac{K_1}{\kappa_1} v_0 dr_1$ in medium 1, and q' by $\frac{K_2}{\kappa_2} v_0 dr_1$ in medium 2, and integrate the four expressions given above, (67) ... (70), with respect to r_1, from 0 to a in medium 1 and from a to b in medium 2. The sum of the integrations performed on (67) and (68) gives the temperature at any point r in medium 1 at any instant t. It will readily be verified that, in virtue of equation (66) above, the expression for the sum of the integration reduces to

$$v = \Sigma \frac{2v_0 b K_2 \sqrt{\kappa_1}}{\lambda r} \frac{\sin a\lambda \sin \mu(b-a)\lambda \sin r\lambda e^{-\kappa_1 \lambda^2 t}}{F(\lambda)}, \quad (71)$$

where

$$F(\lambda) = K_1 \sqrt{\kappa_2} a \sin^2 \mu(b-a)\lambda + K_2 \sqrt{\kappa_1} \mu(b-a) \sin^2 a\lambda - \frac{\sqrt{\kappa_2}}{a\lambda^2}(K_1 - K_2) \sin^2 a\lambda \sin^2 \mu(b-a)\lambda, \quad (72)$$

the summation extending to all the roots of (66). Similarly the sum of the integrations with respect to r_1 performed on (69) and (70) above gives the temperature at r in medium 2 at any instant t: and, as in the case preceding, in virtue of equation (66) above, the expression for the sum of the integrations reduces to

$$v = \Sigma \frac{2v_0 b K_2 \sqrt{\kappa_1}}{\lambda r} \frac{\sin^2 a\lambda \sin \mu(b-r)\lambda e^{-\kappa_1 \lambda^2 t}}{F(\lambda)}, \quad (73)$$

the summation extending to all the roots of (66) as before. These two results contained in (71) and (73) will be found to be in agreement with the solutions of this problem given by Prof. Carslaw in Proc. Cam. Phil. Soc. xx. pp. 400–410 (1920–1).

The problem of determining the temperature at any point in the case where radiation occurs at the outer boundary of the outer material is one of considerable interest, for which the solution has not yet been published. The essential form of the solution is contained in the above paper, but the

problem is worthy of being worked out in full detail, including the two special cases in which the outer material is of greater and of less conductivity respectively than the inner. In like manner the problem of heat conduction in an inner sphere enclosed in a different material extending to infinity requires to be more fully discussed. It would be interesting also, and it would extend the usefulness of the method employed in the present paper to more general problems, to obtain by means of it the solution representing the temperature at any point within a sphere due to a point-source situated at any point within the sphere, the boundary being subject to any specified condition.

I desire to thank James F. Shearer, M.A., B.Sc., of this University for the interest he has taken in the work of this paper.

XXV. *The Propagation of Flame in Gaseous Explosions.* *By* Prof. W. M. Thornton, *D.Sc., D.Eng., Armstrong College, Newcastle-on-Tyne* *.

I. *Introduction.*

THE slow movement of flame in a gaseous explosion distinct from detonation proceeds by activation of gas immediately beyond the wave-front by some emanation from it. As to the nature of this activation, there is at present no general agreement. It can only occur in one of two ways, either by direct ionization from contact with the burning gas, or by changes of energy level in the unburnt molecules. Fifteen years ago the nature of the latter kind of activation had not been worked out, and though there was conclusive evidence that the ignition of gases depended more on the electrical than on the thermal properties of igniting sparks†, theories of ignition had not advanced beyond the conception of this electrical activation as some form of ionization. The proof of change of

* Communicated by the Author.

† "The Electrical Ignition of Gaseous Mixtures," Roy. Soc. Proc. A, xc. p. 272 (1914).

energy levels in atoms and molecules has made possible the view that the activation of gases accompanying ignition either by sparks or flames may not require the pioneering formation of ions, but rather the absorption of radiation in such a manner that their electrical energy is raised, at least for the time necessary to start chemical change. On the other hand Finch and Cowan * have shown recently that in certain modes of ignition by uni directional discharges in gases at lowered pressures the results are proportional to the magnitude of the current, that is, to the rate of passage of ions : but in their work there is the further possibility that in the formation and recombination of ions there is radiation, which may be the active cause in starting the chemical reaction of which flame is evidence.

H. B. Dixon † found that transverse magnetic fields had no influence on the speed of flame in tubes, but the fields used, 10,000 gauss, though technically high, have been regarded since as insufficient to change the movements of ions in the explosion wave-front. Certain experiments by Lindemann and by G. N. Lewis have shown that radiation does not activate molecules over a wide range of frequencies within which it might be expected to occur. The position on this evidence is, then, a deadlock. Two methods of escape seemed possible: on the one hand to test the ionization hypothesis further by experiments in an electric field about the action of which there should be no uncertainty, or to examine the records of flame propagation for evidence which might reveal the mechanism of the process. In his Presidential Address ‡ to Section B of the British Association at Glasgow, Baly discussed very fully the manner in which radiation could enter into chemical reactions, with special reference to photoluminescence. Lenard and Klatt define two "instantaneous" states characteristic of each emission band. "In the upper state, which has a very small temperature range immediately below the upper temperature limit, the stability of the

* S. L. Finch and L. G. Cowan, "Gaseous Combustion in Electrical Discharges. Part II.—The Ignition of Electrolytic Gas by Direct-current Discharge." Roy. Soc. Proc. A, cxvi. p. 529 (1927).

† "Explosion in the Magnetic Field," Roy. Soc. Proc. A, xc. pp. 506-11 (Aug. 1914).

‡ British Association Report, Glasgow, p. 41.

activated state is so small that the whole of the phosphorescent emission takes place within a fraction of a second after activation has ceased." The transfer of energy discussed by Baly is by the formation of complexes formed " not between any two molecules, but only between two which satisfy the conditions, the criterion being that a molecule of one compound, possibly by loss of rotational energy, can give to the molecule of another compound energy equal to the critical quantum of activation of that molecule. A complex of this type may be denoted by the symbol $A^{-}B^{+}$, where B has gained its critical quantum of activation at the expense of the rotational energy of A." Now the most characteristic feature of the combination of molecules in a gaseous explosion is the momentary excess of rotational energy. Evidence for this is the so-called " missing pressure " of explosion *, the percentage of † energy radiated, and the widening of lines in the spectrum of explosions. There is therefore reason for the view that the conditions necessary for the transmission of flame by such an exchange of radiation between members of a complex are all present in the wave-front. There can be no doubt that the action is very limited in space or time, or the velocity of flame, in a tube with an open end for instance, would be much greater, having regard to the normal velocity of molecular movement in gases.

2. *Explosions in an Electric Field of Force.*

Before examining the relation between velocity of explosion and radiation, it may be of interest to give the result of measurements of the velocity of flame in an electric field of force. The influence of a moderate electric field on a steady flame has been examined by Becker ‡. A voltage of 3000 across poles 44 millimetres apart, between which flames of various kinds were maintained, though not making contact with them, has no apparent influence in distorting the flame until metallic salts are introduced.

* "The Lost Pressure in Gaseous Explosions," Phil. Mag., July 1914, p. 18.

† "The Total Radiation from a Gaseous Explosion," Phil. Mag., Sept. 1915, p. 383

‡ A. Becker, "Die Electrischen Eigenshaften der Flamme," *Wien-Harms. Handbuch der Exp. Physik*, xiii. t. 1, p. 117, fig. 1, *c* and *b*.

These are driven by the field towards one pole or the other in a manner shown by the movement of the luminous part of the flame.

The ions which give to flame its electrical conductivity are probably a consequence, and not a cause, of combustion, but they are sufficient partially to short-circuit electrodes between which a strong electric field is applied. It is necessary in such a case to interpose a non-conducting barrier between the two metallic surfaces which form the poles, so that, though it has little influence on the electric gradient, it may prevent a discharge through the gas.

TABLE I.

Upward velocity of flame in 7 per cent. methane in air, central electrode in tube, mean field 10,000 volts per centimetre.

Abs. press., cm. Hg.	Velocity, cm./sec.		Ratio.
	Field on.	Field off.	
74	130	113	1·15
64	130	111	1·17
54	162	132	1·22
44	91	83	1·096
37·5	88	75	1·175
30	61	44	1·38
20	60	44	1·36

A glass tube a metre long and 3 centimetres diameter was surrounded for the middle 80 centimetres with a metal sheath, leaving a space 5 mm. wide through which to observe the passage of the flame. A metal rod 8 mm. diameter, which passed through stoppers at each end, formed the inner pole. Between this and the outer sheath a unidirectional pressure of 10,000 volts was maintained, measured by an electrostatic voltmeter. When the tube was dry, higher pressure could be used, but to avoid any risk of pilot sparks it was kept at the above value. The tube being exhausted and filled with the mixture, which had been standing over water, ignition was made by a spark

at the bottom, and the flame travelled upwards at a uniform rate.

Working with a 9·5 per cent. mixture of methane and air at atmospheric pressure, the lower end of the tube being opened after ignition, no difference in the velocity of the flame could be observed with and without the strongest fields. This observation was made immediately after the publication of Dixon's work, and was held to confirm his conclusions. On repeating this recently with a weaker mixture a transverse electric field was found to have a marked accelerating influence on the speed of the flame.

The velocities are the means of 10 to 20 determinations in each case.

The results are conclusive that at all pressures at and below atmospheric flame travels faster in a transverse electric field.

If the pioneering activation from a wave-front were ionic, the electric field used should have perceptible influence on its rate of transmission, for the velocity of an ion in a field of 10,000 v./cm. is about one-half of the velocity of agitation at normal temperature and pressure. Such a movement as this could not fail to have marked retarding effect on the advance of slow flames if it were solely one of sweeping ions from the wave-front. The observed acceleration by a field may be a consequence of the increase of kinetic energy of the gas, either in or just beyond the wave-front, due to the ionic velocity added by the field, which has the same result as raising the temperature; or, from another point of view, the flame, being a partial conductor, may be drawn up into the field.

The influence of an electric field on radiation from a flame is the widening of the lines of the spectrum in the Stark effect caused by a change of energy of the elliptical orbits of electrons of the vibrating atoms. The Stark effect has the same cause as the transverse Zeeman effect —a modification of the spin of the electrons in the orbit. This change of energy of spin may be the added energy of rotation required by Baly's complex for absorption of radiant energy from combining atoms by atoms of the same kind which are within range. As a possible cause of the accelerating influence of the field this should not be overlooked.

3. *Evidence of Activation by Radiation from Flame.*

Analysis of the slow movement of flame in tubes leads to the view that, when the diameter of the tubes is eliminated from the results, the velocity of transmission is directly proportional to the fourth power of the temperature of the flame. The most general expression for velocity of this uniform slow movement is that of Mallard and Le Chatelier who give *

$$V = \frac{L}{c}\frac{(T-t)}{(t-\theta)}, f(T, t), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where T is the temperature of combustion,

t the ignition temperature of the mixture,

θ the initial temperature,

L the thermal conductivity of the unburnt gas,

c the mean specific heat of the burning or just burnt gas, and

$f(T, t)$ a constant, for a given tube.

This expression has been confirmed by Mason and Wheeler*. Usually θ is atmospheric temperature, and for a given mixture t is constant, so that we may write

$$V = k_1(T-t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where k_1 depends on the size of the tube.

When a flame is advancing with constant speed in a tube each element of the wave-front burns an equal volume of gas in the same time. The area of an annular element of the flame is $2\pi y\,\delta s$, where y is the distance of the boundary from the axis and δs an element of length along the curve. When the gases are perfectly mixed the rate of supply of gas to the flame is uniform over its area projected on to the cross-section of the tube. The projected area of the element is $2\pi y\,\delta y$, and if h is the heat generated per unit area and thickness of this cross-section, the rate of generation of heat in the element is

$$\frac{d}{dt}(2\pi h y\,dy).$$

* See Bone and Townend, 'Flame and Combustion in Gases,' p. 106.
† Trans. Chem. Soc. iii. pp. 1044–1057 (1917).

The velocity of the flame $\frac{dx}{dt}$ is proportional to this, so that

$$\frac{d}{dt}(2\pi h\, y\, dy) = k\frac{dx}{dt},$$

or $$y = \frac{dy}{dx} = \frac{k}{2\pi h}, \text{ a constant.} \quad . \quad . \quad . \quad . \quad (3)$$

The curve for which the subnormal is constant is a parabola, and the flame should therefore assume a parabolic section, as, in fact, it is seen to do, each part moving with the same velocity.

The area of surface of such a flame is

$$\pi\left(\frac{D^3}{3a} + 2aD + \frac{4a^3}{D}\right),$$

where a is the semi-latus rectum. Since these flames are pointed, a is much smaller than D, and the total area is nearly proportional to D^3. Thus the mean heat generated by combustion per unit area and movement of the flame is inversely proportional to the diameter of the tube, for

$$H = \frac{\pi}{4}D^2 h/\pi\left(\frac{D^3}{3a} + 2aD + \frac{4a^3}{D}\right) \doteq \frac{3ah}{4D} \quad . \quad . \quad (4)$$

The corresponding rise of temperature is proportional not only to the mean heat of combustion over the surface of the flame of unit thickness, but to the rate at which the mixture is fed to the flame, that is, to the latter's velocity. Thus $T - t = k_2 V/D$, where k_2 is a constant including $3ah/4$ and other factors from eqn. (1).

Hence

$$D = k_2 V/(T - t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is the working relation between D, V, and $(T - t)$, the mean temperature of the flame being proportional to the ratio of velocity to diameter.

To carry this further, it is necessary to have recourse to observed values of V in tubes of different diameters. Burgess and Wheeler * found the following velocities in a mixture of methane and air giving perfect combustion.

* See Bone and Townend, *loc. cit.* p. 111, fig. 21.

TABLE II.

Diameter of Tube, D.	Velocity of flame, V.	$V/D^{\frac{1}{3}}$.
2·5 cm.	70 cm./sec.	54
5·0	95	55
9·0	105	50
30·5	165	53
96·5	255	55·5
	Mean....	53·5

It follows from their observations that the velocity of uniform motion of flame in tubes is proportional to the *cube root of the diameter of the tube.* The reason for this remarkable result, which seems to have been overlooked, is as follows.

Writing $V=k_3D^{\frac{1}{3}}$, and taking, as above, $V=k_1(T-t)$ we have

$$D=k_4(T-t)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But $D=\dfrac{k_2V}{T-t}$, so that $V=\dfrac{k_4}{k_2}(T-t)^4$, or, nearly, $V=KT^4$.

Thus, when the diameter of the tube is eliminated from the experimental results, the velocity of flame is found to be proportional to the fourth power of its temperature. If this were not so, Burgess and Wheeler's results would not hold.

This, then, is strong evidence that the activation by which the flame travels is radiation according to Stefan's law. The salient facts of the transmission of flame in open tubes are : (i.) it is not retarded by magnetic fields or (ii.) by electric fields, (iii.) the velocity of uniform motion is proportional to the fourth power of the temperature. It is difficult to resist the conclusion that the sole cause of the pioneering activation from flame is electromagnetic radiation.

In the Report of the National Physical Laboratory, 1927, p. 62, evidence is given that flame is in thermal equilibrium with the surrounding vapour according to

black-body radiation. Mallard and Le Chatelier's results show that the velocity of flame is proportional to the total heat generated. The present conclusion is that of this heat only that in the form of radiation is effective in the transmission of flame. For both of these to be true the energy of radiation must bear a constant ratio to the total heat of combustion, and this is known to be the case.

4. *Radiation from Combining Molecules.*

The researches of Hopkinson and David * proved that the heat of combustion radiated to the walls of an explosion vessel was nearly 22 per cent. of the whole. A reason for this figure has been given † as a consequence of the necessary spinning action of two molecules colliding at oblique incidence and cohering. Evidence of such a spinning action is important in the physics of wave-front activation, and since it could only be obtained by spectroscopy, an attempt was made to observe it directly in this manner, the argument being that rotation must, as in other well-known spectroscopic effects, result in widening of lines or bands. A long steel tube was fitted with thick glass windows at each end, with rubber cushions to avoid initial strains. The tube was filled with electrolytic gas, and this ignited by sparking-plugs placed out of sight in T pieces at each end. The flame, as seen through a spectroscope, presented marked differences when advancing towards it or receding. Lines were doubled when seen from the front, but were single, though blurred, when seen from the back, *i.e.*, with the wave-front receding. This observation, it was found later, had been made long ago by Liveing ‡, who attributed the doubling to reversal by absorption of the radiation by the undisturbed atoms of molecules between the wave-front and the window. But the lines are not merely reversed—they are doubled and widened. The so-called "steam lines" in radiation from explosion of undried gases, or from steady flames, have a

* Roy. Soc. Proc. A, lxxxiv. pp. 155–172 (1910), or Hopkinson's Collected Papers.

† "The Lost Pressure in Gaseous Explosions," Phil. Mag., July 1914, p. 18.

‡ "Spectroscopic Studies in Gaseous Explosions," Roy. Soc. Proc. A, xxxvi. pp. 471–78.

similar appearance *. The two chief "steam lines" are nearly equal in intensity, but are widely spaced. The constant occurrence of "reversed" lines in explosion spectra must at least be considered as caused by molecular spin when elements having high translational energy, as all have in gases, collide and combine. It is then necessary to examine the spacing of the steam lines †. From researches on explosion at high pressures Bone concludes that steam is formed at flame temperatures. The temperature of an oxyhydrogen flame is about 2400° C., and the corresponding velocity of hydrogen molecules $6{\cdot}25 \,.\, 10^5$ centimetres per second. The oxygen molecule may be taken to have a diameter of $3 \,.\, 10^{-8}$ cm., the hydrogen molecule $2{\cdot}4 \,.\, 10^{-8}$. When collision takes place, in the accepted planetary manner, and the molecules cohere at contact, the maximum angular velocity of spin given to the resulting steam molecule is :—

$$\omega = 6{\cdot}25 \,.\, 10^5 \Big/ \left(\frac{3+2{\cdot}4}{2}\right) . 10^{-8} = 2{\cdot}3 \,.\, 10^{13} \text{ radians per second,}$$

in effect about the centre of the oxygen molecule. Such a spinning system, carrying its atomic electrons, if at right angles to the line of observation, would give rise to a displacement $\delta\lambda$ of a line of wave-length λ, which is to the left of the spin is one way, to the right if reversed.

Orbits otherwise orientated show the effect less, and if in the line of vision, not at all. This displacement is comparable with the Zeeman effect, and may be tested in the same way. In that case

$$\frac{e}{4\pi mc^2} = \frac{\delta\lambda}{H\lambda^2}, \text{ and } \frac{eH}{2mc} = \frac{\delta\lambda}{\lambda^2} \cdot 2\pi c = \omega \quad . \quad . \quad (5)$$

In the spectra of explosions the steam lines appear at 3064 Å.U. Taking the value of $\omega = 2{\cdot}3 \,.\, 10^{13}$, the displacement $\delta\lambda$ of the lines from a central position is $\omega\lambda^2/2\pi c$, or $11{\cdot}3 \,.\, 10^{-8}$ cm. The total distance between steam lines produced in this manner should be $22{\cdot}6 . 10^{-8}$ cm.

Measurements of the distance apart of the lines taken from Bone and Townend's photographs are not capable of high accuracy, but they are, on an average, a little more

* See Bone and Townend, *loc. cit.* Cp. Pts. xxiv. to xxvii.

† W. A. Bone and others, "Gaseous Combustion at High Pressure," Phil. Trans. Roy. Soc. A, cciv. p. 304.

than a millimetre apart, and 1 millimetre corresponds to $25{\cdot}6 \,.\, 10^{-8}$ cm. These values are so near, having regard to the approximate measurements, that the possibility of the steam lines being formed in this way must be seriously considered. There are no lines in the oxygen or hydrogen spectra at this wave-length, and the origin or cause of the lines, and the reason for their always being found in pairs, very strong and too widely spaced to be simply atomic, as in the case of the sodium lines, has been hitherto unknown. The explanation now suggested is that they are caused by the spin induced by combining collisions of hydrogen and oxygen molecules which lead to the formation of steam, the action being that of skaters approaching at high speed who suddenly link arms. Such a collision, if the bodies were of unequal mass, would give rise to strong oscillations as well as spin, but the translational energy, and therefore the pressure, in the case of a gaseous explosion in which many millions of such combinations occur simultaneously, would certainly be for the moment reduced.

It is possible that the doubled appearance of other than steam lines in the spectra of burning gases may be caused by this mechanical action, which must occur in gases where combining molecules are moving at high speeds before union. The other evidence for such an effect is that the observed pressure of explosion is about one-half of that calculated on the assumption that there is equipartition of energy ; and further, that the total radiation agrees with that which should be available if such a spinning combination took place *. In the calculation of explosion pressures and speeds it is necessary to use specific heats in excess of those found in steady states, but direct evidence for the molecular or atomic movement by which the energy is absorbed is still to be desired. The position is that energy radiated from a flame in a given mixture is a nearly constant proportion of the whole heat of combination ; that the spectroscopic evidence indicates excessive spin in the wave-front ; that the pressure is undoubtedly less than can be accounted for by any thermal exchange without considering the mechanics of the process ; and finally, that by such consideration these facts can be reconciled. An inflammable mixture at the point of ignition can be regarded as a sensitive detector of radiation.

* "The Total Radiation from a Gaseous Explosion," *loc. cit.*, p. 383.

5. *Compression and Ignition.*

Inflammable mixtures heated by sudden compression do not detonate. Flame starts at one point and spreads as from a spark. In Baly's analysis of the conditions under which radiation might become a cause of molecular activation, the excess of vibrational energy required was considered to be available at the expense of the rotational component.

But in ignition by sudden compression there is for a moment an excess of translational energy, and just as it is legitimate to consider "lost" pressure to be a momentary out of balance of equipartition of energy, so in compression the excess pressure can be considered available for such a transfer of vibrational energy in a complex of a reactant molecule and a catalyst, which in the case of gaseous explosions is most generally water-vapour, without which, as H.B. Dixon showed, explosion does not generally occur in gases.

The moment that flame appears, combination proceeds with increasing velocity, and the speed of flame driven forward by the increase of pressure around the source of ignition should be exponential, at least in a closed tube, for each element burnt adds the pressure of its products to force the flame onwards. The records of flame in gases ignited by sparks at the centre of a tube show in every case * a velocity increasing at first exponentially, then reaching a uniform state, and so passing to the end of the tube. To take the case of plate xxii., *loc. cit.*, p. 159, the distances of the flame along the tube from the central point of ignition, at equal arbitrary intervals of time, are expressed by $y = 1{\cdot}242\ e^t$, as shown in Table III.

The velocity dy/dt of the flame at any point y is ty, and the acceleration of the flame from the source is proportional to y, the distance travelled, that is, to *the volume of gas burnt*. This explains the high velocities of explosions in coal-mines, or galleries with a closed end.

Applying this to the spread of flame from a point in a vessel such as the cylinder of an internal combustion engine containing a uniform or turbulent mixture ignited by compression, spark, or hot spot, the acceleration of the flame should be proportional to the cube of the distance from the point at which ignition begins, and the velocity

* See Bone and Townend, plates xi., xii., and xiii.

of flame proportional to the square of the distance moved, until the flame reaches the walls and ends and reflexions occur. In the movement of flame in closed tubes there is, following the period of acceleration, one of uniform velocity. (plates xi. and xii., *supra*); but at the point where the latter begins there is in every case evidence of a wave reflected back to the centre, shown by a line of more intense radiation. The distance at which this first occurs is just beyond half the length from the centre to the closed ends. With open ends this stage is not found. It is caused by compression of the unburnt gas by the expanding products of combustion, and at the transition point the pressures are for the moment equal on both sides of the wave-front.

TABLE III.

t.	e^t.	y. cal.	y. obs.	Velocity of flame in arbitrary units.
0·5	1·648	2·04	2·0	1·0
1·0	2·718	3·35	3·5	3·5
1·5	4·481	5·55	5·5	8·25
2·0	7·389	9·17	9·0	18·0
2·5	12·182	15·1	15·0	37·5
3·0	20·08	24·9	25·0	75·0

Taking V_0 as the volume of the mixture in half the tube, ignition being at the centre, and V that of the unburnt gas, $p(V_0-V)^{\gamma_1}=pV^{\gamma_2}$, where γ_1 is about 1·28 for CO_2, N_2, and OH_2, $\gamma_2=1·35$ for CH_4, N_2, O_2. At equal pressures

$$(V_0-V)^{1·28}=V^{1·35}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from which $V=0·53V_0$, or, with these values of γ, the pressures should be equal a little beyond half the distance between the centre and the end. From plate xi., when there is no initial lag, that is, taking the last two figures, $V/V_0=0·55$. The striæ in photographs of flame are, of course, lines of equal pressure in reflected waves. In the uniform speed stage of explosion there is no acceleration; but pressure equilibrium in the spaces on the two sides of the wave-front.

XXVI. *On the Scattering of α-particles by Light Atoms.* By A. C. Banerji, *M.Sc.(Cal.)*, *M.A.(Cantab.)*, *Head of the Department of Mathematics, Allahabad University* *.

IN a recent Royal Society discussion † on the "Structure of Atomic Nuclei" Prof. Sir Ernest Rutherford sketched out a picture of the gradual building up of atomic nuclei out of protons and electrons. If we suppose that these two are the only primordial elements, we have to find out the fundamental principles underlying the formation of complex nuclei with all their isotopes out of these two. Attempt to achieve this end is being made from several directions :—

(i.) From the experiments of Aston ‡ on the packing fraction of atoms, which give us, in conjunction with Einstein's principle $E=\Delta mc^2$, the amount of energy set free when protons and electrons combine in any way to form atomic nuclei.

(ii.) From study of the origin of γ-rays (Meitner and Ellis) §.

(iii.) From the study of the scattering of α-particles by light atoms (Rutherford, Chadwick) ‖.

The last method is especially interesting. It may be remembered that it was the results of large-angle scattering of α-particles which gave us the nuclear theory of the atom. We give a brief recapitulation of the results.

Let Ze be the central charge, (M, $2e$) be the mass and charge of the α-particle, v_0 is its velocity of projection, p=perpendicular distance of the nucleus from the line of projection. The law of force is supposed to be that of inverse square distance.

Then it is found that when the nucleus is heavy the α-particle describes an hyperbola, with the nucleus as the outer focus. The angle of deviation is $\pi-2\theta$,

where
$$\theta=\tan^{-1}\frac{p}{q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where
$$q=\frac{2Ze^2}{Mv_0^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Communicated by Prof. M. N. Saha, D.Sc., F.R.S.

† Proc. Roy. Soc. cxxiii. p. 373.

‡ *Ibid.* p. 383.

§ Meitner, *Zs. f. Physik*, xxvi. p. 169 ; also *Handbuch der Physik*, xxii. chap. ii. D. Ellis, see recent discussion in Proc. Roy. Soc. cxxiii. p. 385.

‖ Phil. Mag. iv. p. 605 (1927) ; also xlii. (1921) ; l. (1925).

The least distance of approach

$$d = q(1+\sec\theta). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When $p=0, \quad d=2q.$

In actual experiment we are concerned with the total number of particles scattered at an angle θ with the primary beam, and incident upon unit surface placed normally.

The probability of scattering is found to be

$$\frac{1}{r^2}\left(\frac{e^2Z}{Mv_0^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} = \frac{1}{r^2}\frac{(e^2Z)^2}{4E^2}\frac{1}{\sin^4\frac{\theta}{2}}, \quad . \quad . \quad (4)$$

where E = energy of the bombarding α-particle.

If we take an element with the charge Ze, the nearest distance of approach for α-rays of velocity 2×10^9 cm. from ThC′ $= 1{\cdot}71\times10^{-14}$. Z cm. for straight collisions, *i. e.*, when $\theta = 0°$.

The following are the values for a few typical elements for nearest distance of approach for straight collisions :—

$$\text{U}\,(92) = 1{\cdot}56\times10^{-12}\text{ cm.}$$
$$\text{Cu}\,(29) = 5{\cdot}0\times10^{-13}\text{ cm.}$$
$$\text{Mg}\,(12) = 2{\cdot}05\times10^{-13}\text{ cm.}$$
$$\text{He}\,(2) = 3{\cdot}4\times10^{-14}\text{ cm.}$$

It is found that up to copper results of scattering experiments are quite in accord with the formulæ given above ; but marked abnormalities occur in the scattering experiments with light elements. Four elements have been studied in detail, viz., Mg, Al, He, and H. The cases of H and He are a bit more complicated, as a large part of the velocity of bombarding α-particles are communicated to these nuclei. In Al and Mg this factor (communication of velocity to their nuclei) is practically negligible.

The results of these experiments (scattering of α-particles by Al and Mg) have been expressed by Bieler, Chadwick, and Rutherford in the two observed curves (—O—O—O) represented at the end of this paper (figs. 3 and 4, *vide supra*). The ordinate represents the ratio of the observed number of α-particles scattered to the number calculated according to the inverse square law. The abscissa represents the angle of scattering.

Bieler tried to explain these anomalies by assuming that the law of force (repulsive) very close to the nuclei is not given exactly by the inverse square distance, but, in addition, an attractive force comes into play. Bieler assumed that

this attractive force varies as the fourth power of distance, viz. :—

$$F = \frac{2Ze^2}{r^2}\left(1+\frac{r_0}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It should be emphasized that the attractive force is operative only when r is extremely small ($<4\times10^{-13}$ cm.). In heavy atoms up to Cu the least distance of approach is $\geq 5{\cdot}0\times10^{-13}$ cm., hence the classical formulæ hold ; but in light atoms (*e.g.*, H, He) it may be as low as 2×10^{-14} cm., hence the attractive forces make themselves felt.

Working out by means of classical mechanics Bieler * found that if the attractive force varies as the inverse fourth power of the distance, either curve (for Al or Mg) has a horizontal tangent at $\theta=0°$. Thus an inverse fourth-power term in the law of force has no effect on scattering at very small angles. On the other hand, he found that an inverse cube term in the law of force will make itself felt in the scattering no matter how small θ is. But we shall see later on, working out by means of wave-mechanics, that the additional inverse cube force will have also no effect on scattering at very small angles, and that for large angles results obtained from calculations by taking the additional inverse cube force are in good agreement with observed data in the case of Al and Mg.

Debye and Hardemeier † later assumed that the attractive force varied as the inverse fifth power. They argued that as the α-particle approaches the nucleus the intense forces thus arising distort or polarize the constituents of the nucleus. This gives rise to an attractive force on the colliding α-particle, which varies inversely as the fifth power of distance.

Hardemeier ‡ worked out in detail the path of the α-particle under such a law, and also the probability of scattering. He found that with certain assumptions the law can very well explain the experimental results of Bieler. But, as Rutherford § points out, the assumptions are a bit artificial, as the α-particle is supposed to be a point-charge and the nucleus a sphere.

It has recently been supposed that the nucleus has got a spin and a finite magnetic moment. Hence the additional attractive forces are of magnetic origin. The suggestion has not yet been fully worked out.

* Proc. Camb. Phil. Soc. xxi. p. 686 (1923).
† *Phys. Zs.* xxvii. p. 181 (1926).
‡ *Loc. cit.*
§ Discussion, Proc. Roy. Soc. cxxiii. p. 377.

Application of Wave-mechanics.

In recent years the idea has grown up that in the treatment of all atomic phenomena the particle should be replaced by its matter-wave. Born* was the first to apply wave-mechanics to collision problems in general, which includes scattering as a particular case. The α- or β-particle is considered as a de Broglie wave passing through the atom and of length $\lambda = \frac{h}{Mv}$. The velocity of the wave is modified on entering the atom, according to de Broglie's† hypothesis, and scattering of the ψ-function is worked out in very much the same way as diffraction problems are studied in wave-optics. Of the many interesting applications of Born's ideas (*e. g.*, to elastic and non-elastic collisions of electrons with atoms, Ramsauer effect‡, etc.) we confine our attention only to results obtained in the scattering of α-particles. This was first worked out by Wentzel§ assuming the law of inverse square. He obtained a result identical with that obtained in classical mechanics. Sommerfeld‖ has given a different method for calculating the potential inside atoms. He considers that all the outer electrons are assembled in the K-shell, and electricity is distributed according to the law

$$\rho = \frac{eZ^4}{\pi a^3} e^{-2r\frac{Z}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The distribution is the same as we find by calculating the ψ-function of the fundamental orbit of the H-atom according to Schrödinger's theory, and then interpreting the result as a continuous distribution of electricity (*vide* Schrödinger, *Ann. der Phys.* lxxix.). Then V is calculated to be

$$V = 2e^2Z\left(\frac{1}{r} + \frac{Z}{a}\right)e^{-\frac{2rZ}{a}}, \quad . \quad . \quad . \quad . \quad (7)$$

where a is the radius of the hydrogen atom, and the probability of scattering is found to be

$$\frac{1}{r^2}\left(\frac{e^2Z}{Mv^2}\right)^2 \frac{1}{\left(\sin^2\frac{\theta}{2} + \epsilon^2\right)^2}. \quad . \quad . \quad . \quad (8)$$

* *Zs. f. Phys.* xxxvii. p. 863 (1926); xxxviii. p. 803 (1926).
† For modification of the theory see Dirac, *Zs. f. Phys.* xlv.
‡ See Faxen and Holtsmark, *Zs. f. Phys.* xlv. p. 307. Further papers by Holtsmark in *Zs. f. Physik.*
§ Wentzel, *Zs. f. Phys.* xl. p. 590.
‖ Sommerfeld, *Wellen-mechanische Ergänzung*, p. 226.

where $$\alpha = \frac{h}{Mva} \cdot \frac{Z}{2\pi}.$$

As $\alpha \propto 10^{-3}$, its square can be neglected.

Thus Wentzel and Sommerfeld's formula is no improvement upon the classical formula as far as treatment of anomalous scattering is concerned. This is quite apparent if we critically examine Sommerfeld's basic assumption. The extra α^2-term owes its origin to outer ring of electrons which, as far as large angle scattering is concerned, are without effect on the scattering phenomena. The same criticism may be held against another formula by Mitchell *, who calculates the potential by an interesting method due to Fermi †. Sommerfeld himself admits in a footnote that the cases of anomalous scattering are rather to be ascribed to some peculiarity in the nuclear structure.

It will be seen that no attempt has as yet been made to account for the abnormal scattering of α-particles by light atoms with the aid of wave-mechanics. This has been made in the present paper. The first difficulty which was felt was about the choice of a law of force. It was found that if either the inverse fourth-power or the fifth-power law be assumed, the Schrödinger wave-equation for the motion of any charged particle round the nucleus has no solution in polynomials with finite number of terms, giving discrete energy-values. But it is quite certain that such discrete values must exist, as otherwise it will not be possible to explain the origin of the γ-ray spectrum, which, according to the investigations of Meitner ‡, Ellis ‡, and Kuhn §, arise from the transition of the α-particle between levels having discrete energy-values. Again, for the scattering the Schrödinger equation gives us a solution when the attractive force varies inversely as the cube of the distance. For the inverse fourth-power law or the fifth-power law the solution will depend on the integral $\int_0^\infty \frac{e^{bx}}{x}\,dx$ (b being a constant), which does not converge.

I have therefore worked out the scattering formula by wave-mechanics by assuming the inverse cube law.

The path of the α-particle round the nucleus has been worked out by means of classical mechanics. In a second

* Mitchell, Proc. Nat. Acad. Sci. Washington (1929).
† Fermi, *Zs. f. Phys.* xlviii. p. 73.
‡ *Loc. cit.*; also Proc. Camb. Phil. Soc. xxii. p. 844 (1925).
§ *Zs. f. Phys.* xliii. p. 56; xliv. p. 32.

paper I have worked out the theory of the γ-ray spectrum on the basis of the inverse cube law.

Path of the α-particle round the nucleus worked according to the law of force

$$F = \frac{2e^2Z}{r^2}\left(1 - \frac{r_0}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad (5')$$

where r_0 is the distance at which force changes in sign, *i. e.*, where its value is zero.

Let v_0 be the velocity of projection at infinity, and p be the perpendicular from the origin on the asymptote, then the equation to find the orbit is

$$\frac{d^2u}{d\theta^2} + u\left(1 - \frac{2Ze^2r_0}{Mv_0^2p^2}\right) = \frac{-2Ze^2}{Mv_0^2p^2}, \quad . \quad . \quad . \quad (8)$$

where $$u = \frac{1}{r}.$$

Put $$q \,.\, Mv_0^2 = 2Ze^2;$$

we get

$$\frac{d^2u}{d\theta^2} + u\left(1 - \frac{qr_0}{p^2}\right) = -\frac{q}{p^2}. \quad . \quad . \quad . \quad . \quad (8')$$

We shall measure θ from the line of minimum (apsidal) distance. Let x be the angle which the asymptote makes with the initial line.

Case I. (*vide* fig. 1) :

When $p^2 > qr_0$,

put $$n^2 = 1 - \frac{qr_0}{p^2};$$

we get $$\tan nx = \frac{pn}{q},$$

and $$\left.\begin{aligned} \frac{1}{r} &= -\frac{q}{p^2 - qr_0} + \frac{\sqrt{p^2 + q^2 - qr_0}}{p^2 - qr_0}\cos n\theta. \\ \text{Minimum distance } d \text{ is given by} \\ d &= \frac{p^2 - qr_0}{\sqrt{q^2 + p^2 - qr_0} - q}. \end{aligned}\right\} \quad . \quad . \quad (9\,\text{A})$$

Case II.:

When $p^2 < qr_0$,

put $$n'^2 = \frac{qr_0}{p^2} - 1;$$

we get
$$\tanh n'x = \frac{pn'}{q},$$

and
$$\frac{1}{r} = \frac{q}{qr_0 - p^2} - \frac{\sqrt{q^2 - (qr_0 - p^2)}}{qr_0 - p^2} \cosh n'\theta,$$

and
$$d = \frac{qr_0 - p^2}{q - \sqrt{q^2 - (qr_0 - p^2)}}. \qquad (9\,\text{B})$$

In order that d shall be real

$$q^2 > qr_0 - p^2,$$

or
$$p^2 > qr_0 - q^2.$$

This condition is also necessary that $\tanh n'x < 1$, i. e., x may be real.

Case III. :

When $p^2 = qr_0$.

The equation to the orbit is

$$\frac{1}{r} = \frac{1}{2q} - \frac{\theta^2}{2r_0},$$

also
$$x = \frac{r_0}{p} = \sqrt{\frac{r_0}{q}}, \qquad (9\,\text{C})$$

$$d = 2q.$$

Case IV. :

When $p^2 \leq qr_0 - q^2$ and $r_0 > q$.

No real values of d and x are possible, and the α-particle falls into the nucleus.

In this case, taking the initial line parallel to the asymptote or the initial direction of motion, the equation to the orbit is

$$\frac{1}{r} = \frac{q}{qr_0 - p^2}(1 - \cosh n'\theta) - \frac{1}{pn'} \sinh n'\theta \qquad (9\,\text{D})$$

If $\frac{\pi}{2} > x > 0$, the α-particle describes an asymptotic curve proceeding back to infinity without going round the nucleus —the deflexion is given by $\pi - 2x$.

If $\pi > x > \frac{\pi}{2}$, the particle goes round the nucleus and proceeds back to infinity—the deflexion is given by $2x - \pi$.

For the region $p^2 > qr_0$, x has a *minimum* value x_m at $p=p_m$, where p_m satisfies the equation

$$\tan \frac{p_m^2 \sqrt{p_m^2 - r_0}}{r_0(q^2 + p_m^2 - qr_0)} = \frac{\sqrt{p_m^2 - qr_0}}{q}. \quad . \quad (10\,\text{A})$$

This gives a *maximum* deflexion.

For the region $p^2 < qr_0$, x has a *maximum* value x_n at $p=p_n$, where p_n satisfies the equation

$$\tanh \frac{p_n^2 \sqrt{qr_0 - p_n^2}}{r_0(q^2 + p_n^2 - pr_0)} = \frac{\sqrt{qr_0 - p_n^2}}{q}. \quad . \quad . \quad (10\,\text{B})$$

This also gives a *maximum* deflexion.

We have seen that when $p^2 = qr_0$,

$$x = x_0 = \sqrt{\frac{r_0}{q}};$$

at this value of p, x decreases as p increases, and *vice versa*.

Case A.:

$$\pi > x_0 > \frac{\pi}{2}.$$

For the region $p > p_1$, where $p_1 > \sqrt{qr_0}$ and p_1 is given by

$$\tan \sqrt{p_1^2 - qr_0} \cdot \frac{\pi}{2p_1} = \frac{1}{q} \sqrt{p_1^2 - qr_0}, \quad . \quad . \quad (11\,\text{A})$$

the α-particle returns back to infinity without going round the nucleus.

For the region $p < p_1$, provided $r_0 \leq q$, otherwise for the region $\sqrt{qr_0 - q^2} < p < p_1$, provided $r_0 < q$, the α-particle goes round the nucleus and returns back to infinity.

For the region $p \leq \sqrt{qr_0 - q^2}$, provided $r_0 > q$, the α-particle falls into the nucleus.

Case B.:

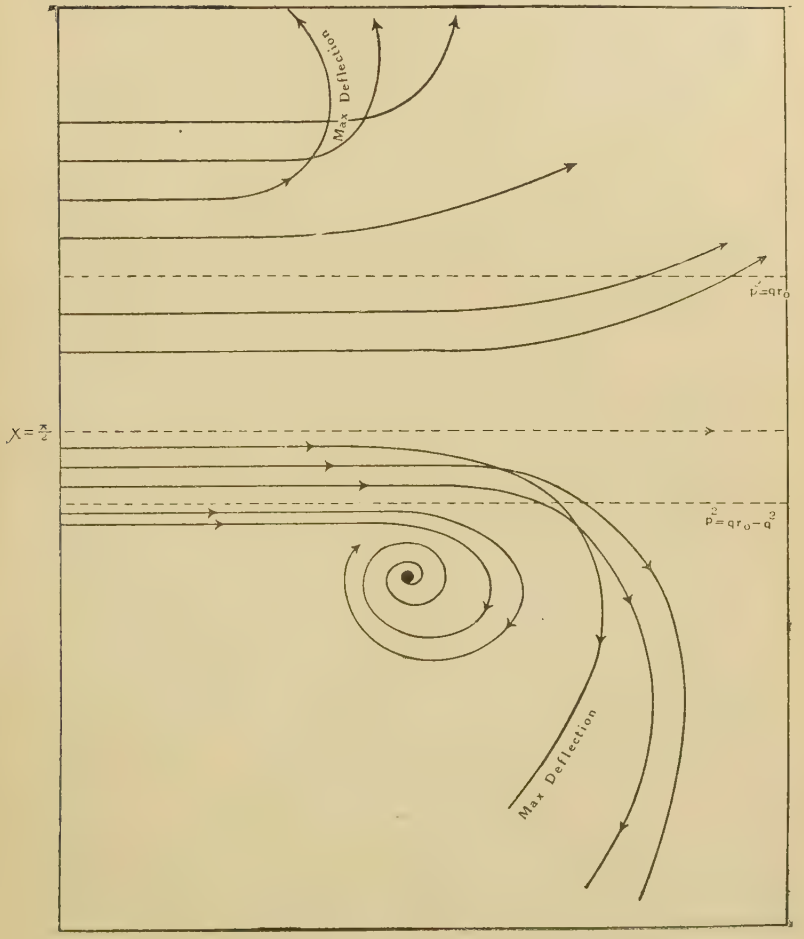
$$0 < x_0 < \frac{\pi}{2}.$$

For the region $p > p_2$, where $p_2 < \sqrt{qr_0}$, and p_2 is given by

$$\tanh \sqrt{qr_0 - p_2^2} \cdot \frac{\pi}{2p_2} = \frac{1}{q} \sqrt{qr_0 - p_2^2}, \quad . \quad . \quad (11\,\text{B})$$

provided $x_n > \frac{\pi}{2}$, the α-particle returns back to infinity without going round the nucleus.

Fig. 1.



Actual diagram of orbits calculated according to the law

$$F=\frac{2Ze^2}{r^2}\left(1-\frac{r_0}{r}\right).$$

The figures have been drawn for the case $r_0=2q$, where $q=\frac{2Ze^2}{Mv_0^2}$. The three regions, (1) $p^2>qr_0$, (2) $p^2<qr_0$ but $>qr_0-q^2$, are shown by the two dotted lines. The intermediate dotted line, with arrow, is the line of demarcation between orbits going round the nucleus and orbits not going round the nucleus. The spirals inside $p^2=qr_0-q^2$ show the path taken by the α-particle as it approaches the nucleus (shown by the heavy dot), and causes artificial disintegration by ultimately falling into it.

For the region $p < p_2$, provided $r_0 \leq q$, otherwise for the region $\sqrt{qr_0 - q^2} < p < p_2$, provided $r_0 > q$ and $p_2 > \sqrt{qr_0 - q^2}$, the α-particle goes round the nucleus and returns back to infinity.

As before, for the region $p \leq \sqrt{qr_0 - q^2}$, where $r_0 > q$, the α-particle falls into the nucleus.

If $x_n < \frac{\pi}{2}$, or if $p_2 < \sqrt{qr_0 - q^2}$, the α-particle never goes round the nucleus.

It is to be noticed that $q \propto Z$ the atomic number. If we also suppose that the attractive inverse cube-law force has magnetic origin, then r_0 varies as $\frac{1}{Z}$ roughly, because the resultant magnetic moment of the nucleus will be the vector sum of the individual magnetic moments of the constituent particles. For artificial disintegration of the nucleus the α-particle must fall into the nucleus and shatter it. For this r_0 must be $> q$. So we find that the lighter atoms for which $r_0 > q$ can only be artificially disintegrated.

Protons are released from the nucleus both in the forward and backward directions. It has been suggested by Rutherford and Andrade * that, when the release is in the forward direction, the α-particle does not go round the nucleus, but when the release is in the backward direction the α-particle goes round the nucleus. This is in agreement with the mathematical result obtained above.

We now treat the problem of scattering by wave-mechanics, taking the potential to be given by

$$V = 2e^2Z\left(\frac{1}{r} - \frac{\beta}{r^2}\right)e^{-\frac{2rZ}{a}}, \quad . \quad . \quad . \quad . \quad (7')$$

where $\beta = \frac{r_0}{2}$, which is small, and, we shall see later, is of the order 10^{-14} cm., and "a," the radius of the hydrogen atom, is of the order 10^{-8} cm.

Potential vanishes at

$$r = \beta.$$

The law deviates appreciably from that of inverse square law when r is of the order 10^{-13} cm. or less, Z can never be larger than 100. Therefore the exponential factor is at most of the order $e^{-1/1000}$, which differs little from unity.

Outside the atomic radius the potential falls rapidly, and ultimately V will be negligible.

* Andrade, 'The Structure of Atoms,' p. 99.

We follow Born's method, using Sommerfeld's notation. The wave equation for the total system is

$$\nabla^2\psi + \frac{8\pi^2 M}{h^2}(E-V)\psi = 0. \quad . \quad . \quad . \quad (12)$$

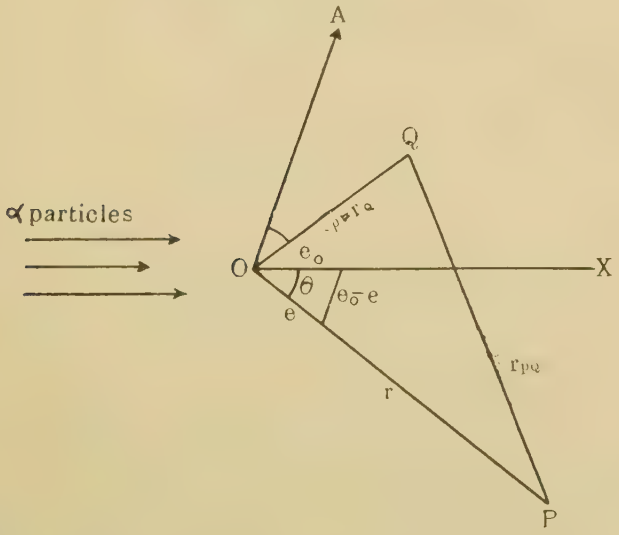
Put $$k^2 = \frac{8\pi^2 ME}{h^2},$$

and write $$\psi = \psi_0 + \psi_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where ψ_0 is due to the on-coming plane wave of α-particles, and ψ_1 is due to scattering and has the spherical wave-form.

ψ_1 is small compared with ψ_0, and also $V\psi_0$ is small in comparison with ψ_0 and $\nabla^2\psi_0$.

Fig. 2.



We get the two equations

$$\left.\begin{aligned} \nabla^2\psi_0 + k^2\psi_0 &= 0, \\ \nabla^2\psi_1 + k^2\psi_1 &= \frac{k^2 V}{E}\psi_0. \end{aligned}\right\} \quad . \quad . \quad . \quad (12\text{A})$$

We obtain the solution

$$\left.\begin{aligned} \psi_0 &= e^{ikz_Q}, \\ \psi_1 &= -\frac{k^2}{4\pi E}\int V(r_Q)\frac{e^{ikr_{PQ}}}{r_{PQ}}e^{ikr_Q}\,d\tau; \end{aligned}\right\} \quad . \quad (14)$$

where "O" is the centre of the atom, the element of volume $d\tau$ is situated at Q at a distance r_Q from "O", scattering is observed at P at a distance r from "O".

OX is drawn parallel to the motion of the on-coming α-particles.

Take e and e_0 as unit vectors in the directions of r and x, r is large compared with r_Q, which is of the order of atomic radius.

As a first approximation,

$$r_{PQ} = r - (e, r_Q),$$

$$x_Q = (e_0, r_Q).$$

Hence

$$\psi_1 = -\frac{k^2}{4\pi E}\frac{e^{ikr}}{r}\int V(r_Q)\left\{1 + \frac{(e, r_Q)}{r}\right\}e^{ik(e_0-e,\, r_Q)}.d\tau. \quad (15)$$

Now take $r_Q = \rho$, and also we have

$$(e_0 - e, r_Q) = 2\rho \sin\frac{\theta}{2}\cos\phi,$$

$$(e, r_Q) = \rho \sin\left(\phi - \frac{\theta}{2}\right),$$

where

$$\angle XOP = \theta \quad \text{and} \quad \angle AOQ = \phi,$$

OA being drawn parallel to vector $e_0 - e$.

$$\therefore \quad \psi_1 = -\frac{k^2}{4\pi E}\frac{e^{ikr}}{r}\int V(r_Q) \times\left\{1 + \frac{\rho}{r}\sin\left(\phi - \frac{\theta}{2}\right)\right\}e^{i\,2k\rho\sin\frac{\theta}{2}\cos\phi}.d\tau.$$

Take ρ, ϕ, χ as polar coordinates.

As potential becomes practically negligible at a distance greater than the atomic radius, the integral can be taken throughout all space without any serious error.

Substituting the value of $V(r_Q)$ and putting $r_Q = \rho$, we get

$$\psi_1 = -\frac{k^2e^2Z}{2\pi E}\frac{e^{ikr}}{r}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{1}{\rho} - \frac{\beta}{\rho^2}\right) \times\left\{1 + \frac{\rho}{r}\sin\left(\phi - \frac{\theta}{2}\right)\right\}e^{-\frac{2Z}{a}\rho + i2k\sin\frac{\theta}{2}\cos\phi\,.\,\rho}.\rho^2\sin\phi\, d\rho\, d\phi\, \chi.$$

Put $$c = \frac{2Z}{a}; \qquad d = 2k\sin\frac{\theta}{2}.$$

Then

$$\psi_1 = \frac{-k^2e^2Z}{E}\cdot\frac{e^{ikr}}{r}\int_0^{\pi}\sin\phi\,d\phi\int_0^{\infty}(\rho-\beta) \times\left\{1+\frac{\rho}{r}\sin\left(\phi-\frac{\theta}{2}\right)\right\}e^{-(\sigma-id\cos\phi)\cdot\rho}\cdot d\rho.$$

Integrating, we get after some work and putting

$$k=\frac{2\pi}{\lambda},\qquad \lambda=\frac{h}{MV},\qquad \alpha=\frac{\lambda}{a}\frac{Z}{2\pi},$$

and remembering that

λ is of the order 10^{-13} cm.,
α is of the order 10^{-3},
β is of the order 10^{-13} cm.,
r is finite,

$$\psi_1=\frac{-e^2Z}{2E}\cdot\frac{e^{i\frac{2\pi}{\lambda}r}}{r}\left[\frac{1}{\sin^2\frac{\theta}{2}}-\frac{4\pi\beta}{\lambda}\frac{\tan^{-1}\left(\frac{2a\pi}{Z\lambda}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}\right] \quad . \quad . \quad . \quad (15\,A)$$

$$\therefore\ |\psi_1|=\left(\frac{e^2Z}{2Er}\right)\left|\frac{1}{\sin^2\frac{\theta}{2}}-\frac{4\pi\beta}{\lambda}\frac{\tan^{-1}\left(\frac{2a\pi}{Z\lambda}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}\right|.$$

Also $$|\psi_0|=1.$$

If W is the probability of scattering, then

$$W=\left|\frac{\psi_1}{\psi_0}\right|^2 =\left(\frac{e^2Z}{2Er}\right)^2\left\{\frac{1}{\sin^2\frac{\theta}{2}}-\frac{4\pi\beta}{\lambda}\frac{\tan^{-1}\left(\frac{2a\pi}{Z\lambda}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}\right\}^2. \quad (16)$$

Let W_0 be the probability of scattering according to the inverse square law, then

$$W_0=\left(\frac{e^2Z}{2Er}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}.$$

$$\therefore \quad \frac{W}{W_0} = \left\{1 - \frac{4\pi\beta}{\lambda}\tan^{-1}\left(\frac{2\pi a}{\lambda Z}\sin\frac{\theta}{2}\right).\sin\frac{\theta}{2}\right\}^2. \quad (17)$$

We draw a curve, taking $\frac{W}{W_0}$ as ordinate and θ as abscissa.

It is clear that at $\theta = 0°$ the curve has the horizontal tangent $\frac{W}{W_0} = 1$. So we find that the additional inverse cube power term in the law of force has practically no effect on the scattering at very small angles, contrary to what Bieler thought.

Now

$$\frac{2\pi a}{\lambda Z} > \frac{2\pi \times 2{\cdot}4 \times 10^{-8}}{6 \times 10^{-13} \times 10^2} > 2{\cdot}5 \times 10^3,$$

as Z is never greater than 100.

So if θ is not very small ($> 3°$), we have

$$\tan^{-1}\left(\frac{2\pi a}{\lambda Z}\sin\frac{\theta}{2}\right) = \frac{\pi}{2}, \qquad \text{approx. } (> 89°).$$

We can write

$$\frac{W}{W_0} = \left(1 - \frac{2\pi^2\beta}{\lambda}\sin\frac{\theta}{2}\right)^2. \quad . \quad . \quad . \quad (17\,\text{A})$$

Scattering of α-particles by Aluminium *.

α-particles from radium C of mean range 6·6 cm. were made to bombard aluminium nucleus. The curve obtained by Bieler from experimental data may approximately be represented by

$$\frac{W}{W_0} = \left(1 - {\cdot}25\,\sin\frac{\theta}{2}\right)^2. \quad . \quad . \quad . \quad (17\,\text{B})$$

The curves given by the above formula, as well as the curve obtained from actual data by Bieler, are drawn together for the sake of comparison. We give below some of the calculated values obtained from the formula :—

$\theta = 100°$,	$\frac{W}{W_0} = {\cdot}66$;	$\theta = 70°$,	$\frac{W}{W_0} = {\cdot}74$;
$\theta = 90°$,	$\frac{W}{W_0} = {\cdot}68$;	$\theta = 60°$,	$\frac{W}{W_0} = {\cdot}77$;
$\theta = 80°$,	$\frac{W}{W_0} = {\cdot}705$;	$\theta = 40°$,	$\frac{W}{W_0} = {\cdot}84$.

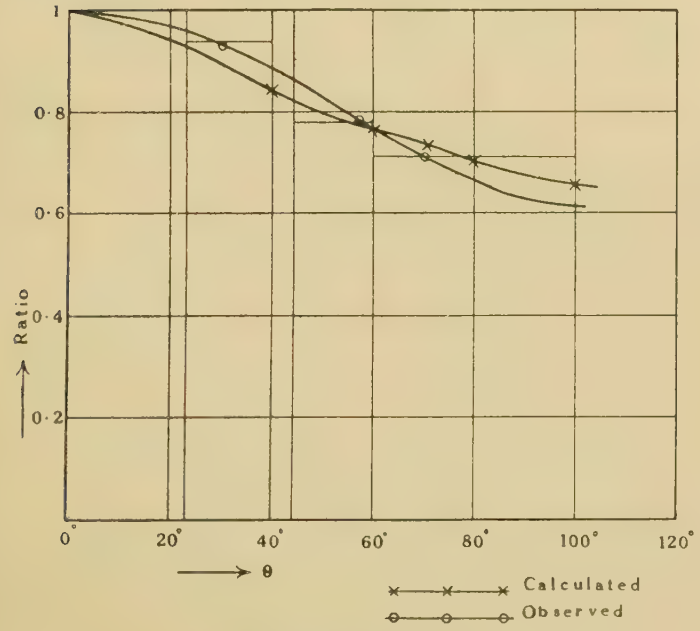
* For observed data see Bieler, Proc. Roy. Soc. cv. p. 445 (1924).

Now for range 6·6 cm. the velocity is $1{\cdot}89 \times 10^9$ cm.

$$\therefore \quad \lambda = \frac{h}{Mv} = \frac{6{\cdot}55 \times 10^{-27}}{6{\cdot}56 \times 10^{-24} \times 1{\cdot}89 \times 10^9} \text{ cm.} = 5{\cdot}3 \times 10^{-13} \text{ cm.}$$

$$r_0 = 2\beta = \frac{2 \times {\cdot}127\,\lambda}{\pi^2} = 1{\cdot}4 \times 10^{-14} \text{ cm.}$$

Fig. 3.



Scattering by Aluminium.

The ordinate represents the ratio $\frac{W}{W_0}$, W = observed or calculated scattering, W_0 = scattering calculated according to inverse square law.

This seems to be of the right order in dimension. The effect of the attractive force will be appreciable at 10 times or 20 times this distance. This force may have magnetic origin. At distance $10r_0$ it will be 1/10 of the electrostatic force and at distance $20r_0$ it will be 1/20 of the electrostatic force, and so on; if this additional attractive

force has magnetic origin, then βZ will be constant—*i. e.*, $r_0 \propto \frac{1}{Z}$.

Moreover, β or r_0 should also vary with the velocity of the bombarding α-particle.

Scattering of α-particles by Magnesium *.

α-particles from Ra C of mean range 6·2 cm. are made to bombard on magnesium. The curve obtained may be approximately represented by

$$\frac{W}{W_0} = \left(1 - \cdot 26 \sin \frac{\theta}{2}\right)^2. \quad . \quad . \quad . \quad (17\,c)$$

The curve given by the above formula, as well as the curve obtained by Bieler from actual data, are drawn together for the sake of comparison.

We give below some of the calculated values :—

$\theta = 100°$, $\frac{W}{W_0} = \cdot 64$; $\theta = 70°$, $\frac{W}{W_0} = \cdot 73$;

$\theta = 90°$, $\frac{W}{W_0} = \cdot 67$; $\theta = 60°$, $\frac{W}{W_0} = \cdot 76$;

$\theta = 80°$, $\frac{W}{W_0} = \cdot 71$; $\theta = 40°$, $\frac{W}{W_0} = \cdot 83$.

Now for range 6·2 cm. $v = 1 \cdot 84 \times 10^9$ cm.

$$\therefore \quad \lambda = \frac{6 \cdot 55 \times 10^{-27}}{6 \cdot 56 \times 10^{-24} \times 1 \cdot 84 \times 10^9} \text{ cm.} = 5 \cdot 4 \times 10^{-13} \text{ cm.}$$

$$\therefore \quad r_0 = 2\beta = \frac{2 \times \cdot 13\lambda}{\pi^2} = 1 \cdot 5 \times 10^{-14} \text{ cm.}$$

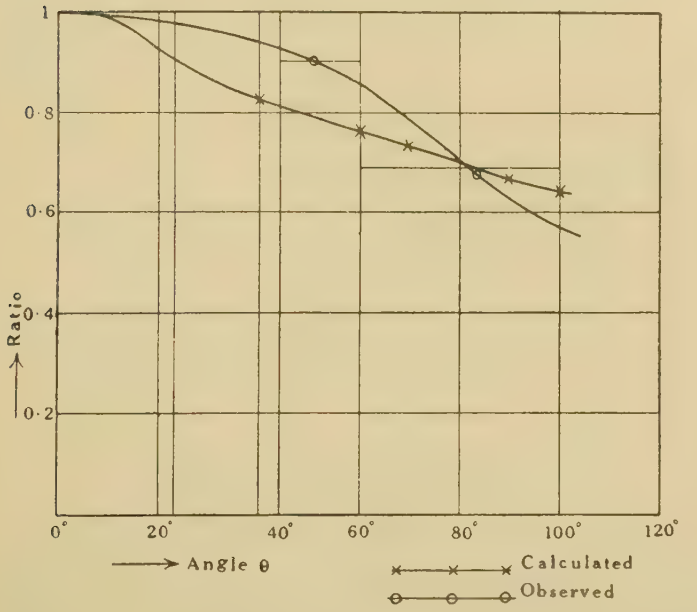
Both in the case of magnesium and aluminium we have neglected the motion of the nucleus, as even for such a large angle of scattering as 100° the correction for this is only 1·5 per cent. This is considerably less than the experimental error involved.

Bieler's data for aluminium and magnesium do not seem to be quite satisfactory. For α-particles of mean range 6·6 cm. bombarding aluminium nucleus the ratio $\frac{W}{W_0}$ between 23°·7

* For observed data see Bieler, Proc. Roy Soc. cv. p. 445 (1924).

and 44°·2 was found to be $0{\cdot}94 \pm 0{\cdot}02$, and this ratio between 43°·9 and 59°·7 was found to be $0{\cdot}78 \pm 0{\cdot}02$. So that at 44° it is not clear whether this ratio has the value 0·92 or 0·8. Again, for α-particles of mean range 3·0 cm. bombarding magnesium nucleus the ratio $\frac{W}{W_0}$ between 43°·9 and 59°·7

Fig. 4.



Scattering by Magnesium.

The ordinate represents the ratio $\frac{W}{W_0}$, W = observed or calculated scattering, W_0 = scattering calculated according to inverse square law.

has the value $1{\cdot}05 \pm 0{\cdot}06$, whereas between 60°·3 and 99°·5 this ratio has the value $0{\cdot}79 \pm 0{\cdot}08$. It seems therefore that the decrease in the value of $\frac{W}{W_0}$ at 60° is too abrupt. It seems desirable that the experiments should be repeated

There is some experimental evidence that r_0 varies as the velocity of bombarding α-particles.

α-particles from polonium of mean range 3 cm. are made to bombard magnesium nucleus. In this case our curve can be roughly represented by

$$\frac{W}{W_0}=\left(1-\cdot 18\sin\frac{\theta}{2}\right)^2. \quad . \quad . \quad . \quad (17\,\text{D})$$

We write some calculated values :

$$\theta=90^\circ, \quad \frac{W}{W_0}=\cdot 76\ ;$$

$$\theta=80^\circ, \quad \frac{W}{W_0}=\cdot 79.$$

For range 3·0 cm. $v=1\cdot 45\times 10^9$ cm.

$$\therefore \quad \lambda=\frac{6\cdot 55\times 10^{-27}}{6\cdot 56\times 10^{-24}\times 1\cdot 45\times 10^9}\ \text{cm.}=6\cdot 9\times 10^{-13}\,\text{cm.}$$

$$\therefore \quad r_0=2\beta=\frac{\cdot 18\lambda}{\pi^2}=\frac{\cdot 18\times 6\cdot 9\times 10^{-13}}{9\cdot 93}=1\cdot 251\times 0^{-14}\,\text{cm.}$$

$$\therefore \quad \frac{(r_0)_{\text{3 cm. range}}}{(r_0)_{\text{6·2 cm. range}}}=\frac{1\cdot 25}{1\cdot 5}=\cdot 83.$$

Also $$\frac{(v_0)_{\text{3 cm. range}}}{(v_0)_{\text{6·2 cm. range}}}=\frac{1\cdot 45}{1\cdot 84}=\cdot 79.$$

$\therefore$ r_0 varies roughly as v for the same element.

This may be taken as some experimental evidence for the magnetic origin of the forces, because the ponderomotive force on a charged particle in a magnetic field varies as the velocity of the particle.

Scattering as a function of energy of α-particles :

In fig. 5 is represented the curves in which $\frac{W}{W_0}$ is plotted against $\frac{1}{E}$ for aluminium and magnesium, for a given angle of scattering ($\theta=135^\circ$). Our formula can now be re-written as a function of $\frac{1}{E}$. We get

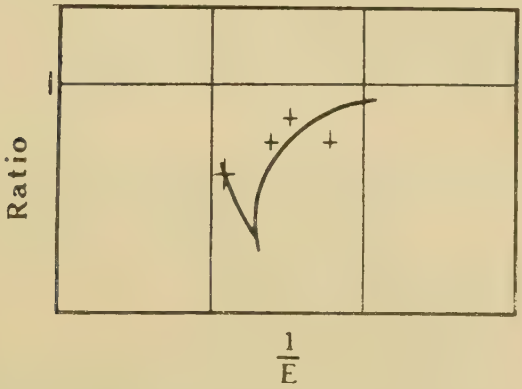
$$\frac{W}{W_0}=\left(1-\frac{C}{x^{\frac{1}{2}}}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (17\,\text{E})$$

where $$x = \frac{1}{E},$$

and $$C = \frac{2\pi^2\beta}{h}\sqrt{2M}.$$

The formula can be made to represent the curve on either side of the minimum, but there is discrepancy near about the minimum. The minimum value, according to formula, of $\frac{W}{W_0}$ is 0, whereas according to experimental results the minimum is about ·3.

Fig 5.



It is impossible to remove this discrepancy with whatever law of force we may work, since $\left|\frac{\psi_1}{\psi_0}\right|^2$ will always occur as a square and will have the minimum value zero. Hence the discrepancy is to be ascribed to our use of the Schrödinger electron.

It is now admitted that since the Schrödinger form of the wave-equation of the electron does not give the proper relativity correction, and fails also to take into account the magnetic properties of the electron (rotating electron), it should be replaced by a better model. At the present time Dirac's model of the electron and wave-equation is holding the field. One of the features of this theory is the introduction of four ψ-functions in place of one, and in normalizing we have to take the sum of their squares instead of the

square of a single ψ. The Dirac equation for the α-particle or any positive particle has not yet been framed. Attempts are being made in this direction.

The cases of hydrogen and helium have to be treated differently in a separate paper. In the case of hydrogen nucleus we observed the disintegrated hydrogen particles instead of scattered α-particles, and in the case of helium we observe both the protons from helium nucleus as well as scattered α-particles. Moreover, in these two cases we cannot neglect the motion of the nucleus.

I have great pleasure in thanking my colleague, Prof. M. N. Saha, F.R.S., of Allahabad University, for much help and many valuable suggestions in preparing this paper.

Mathematics Department,
University of Allahabad,
November 20, 1929

XXVII. *The Ionization-Formula and the New Statistics.* *By* S. Chandrasekhar *.

(1) *Introduction.*

THE derivation of the formula

$$\frac{n_+ \cdot n_-}{n} = \frac{1}{h^3}(2\pi mk\mathrm{T})^{3/2} \cdot e^{-\chi/kt} \cdot \sigma .$$

(n_+ = the number of ionized atoms,
n_- = the number of electrons,
n = the number of neutral atoms)

has attracted much attention after its first derivation by Saha. The rigorous way of deriving the above equation has been successfully attempted by Fowler † ; but no attention has so far been given to modifying the above formula on the new statistics of Fermi-Dirac. Obviously it will be of no practical importance if in the application of the new statistics we treat all the constituents—the atoms of both kinds and

* Communicated by the Author.

† R. H. Fowler, Phil. Mag. xlv. p. 1 (1923); also Fowler and Milne, 'Monthly Notices,' R. A. S. lxxxiii. p. 407 (1923). A simple way of deriving the equation (1) has been given by W. F. G. Swann, Journ. of Frank. Inst. cc. p. 591 (1925).

the electrons—as degenerate, because the ionization formula, being most applied to the thermodynamics of a star, would not justify our assumption as to the degeneracy of the atoms themselves. But the electrons, even at that high temperature, if the pressure is sufficiently high, will be a degenerate system *. Thus Sommerfeld's † condition of degeneracy,

$$\frac{nh^3}{2} \cdot \frac{1}{(2\pi mk\mathrm{T})^{3/2}} > 1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

gives, if the number of electrons per c.c. is 10^{30}, the system to be degenerate up to temperatures below $3{\cdot}5^{\circ} \times 10^9$, and if $n_- = 10^{27}$ the system will be degenerate even up to a temperature of $3^{\circ} \times 10^7$.

The object of this paper is to modify the ionization formula on the Fermi-Dirac statistics for the degeneracy of the electrons. But, before dealing with that problem, the case of complete degeneracy will be discussed. The modified formula when the degeneracy of the electrons alone is considered will follow naturally from the case of complete degeneracy.

(2) *General Relation between n_+, n, and n_-.*

According to the new statistics of Fermi-Dirac, the entropy of a system is given by ‡

$$\mathrm{S} = nk\left[\frac{5}{2}\frac{\mathrm{U}_{3/2}}{\mathrm{U}_{1/2}} - \log \mathrm{A}\right], \quad . \quad . \quad . \quad . \quad (3)$$

where $\mathrm{U}_{3/2}$ and $\mathrm{U}_{1/2}$ are the special cases of the general Sommerfeld § integral

$$\mathrm{U}_\rho = \frac{1}{\Gamma(\rho+1)}\int_0^\infty \frac{u^\rho du}{\frac{1}{\mathrm{A}}e^u + 1} \cdot \quad . \quad . \quad . \quad . \quad (4)$$

For a mixture of gases the entropy is given by

$$\mathrm{S} = \Sigma\, kn_\nu\left[\frac{5}{2}\frac{\mathrm{U}_{3/2}}{\mathrm{U}_{1/2}} - \log \mathrm{A}_\nu\right]. \quad . \quad . \quad . \quad (5)$$

In the case under consideration, that of a monatomic gas dissociating into positive ions and electrons, the summation comprises these terms.

* In this connexion, see an interesting contribution by Dr. E. C. Stoner, Phil. Mag. vii. p. 63 (1929).
Sommerfeld, *Zeits. f. Phys.* xlvii. p. 1 (1928).
‡ Fermi, *Zeits. f. Phys.* xxxvi. p. 902 (1926).
§ Sommerfeld, *Zeits. f. Phys.* xlvii. p. 1 (1928).

If δn represents an arbitrary change in the number of neutral atoms, δn_+ and δn_- the corresponding changes for the positive ions and the electrons, the corresponding change in the entropy is given by

$$\delta S = \Sigma \frac{\partial}{\partial n}\left[nk\left\{\frac{5}{2}\frac{U_{3/2}}{U_{1/2}} - \log A\right\}\right]$$

$$= \Sigma\left\{k\left[\frac{5}{2}\frac{U_{3/2}}{U_{1/2}} - \log A\right] + nk\frac{\partial}{\partial A}\left[\frac{5}{2}\frac{U_{3/2}}{U_{1/2}} - \log A\right] \times \frac{\partial A}{\partial n}\right\}\delta n. \quad . \quad (6)$$

Now

$$\frac{\partial}{\partial A}\left[\frac{5}{2}\frac{U_{3/2}}{U_{1/2}} - \log A\right] = \left\{\frac{5}{2}\left[\frac{U'_{3/2}}{U_{1/2}} - \frac{U_{3/2}\,.\,U'_{1/2}}{U^2_{1/2}}\right] - \frac{1}{A}\right\}. \quad (7)$$

Now

$$U_\rho' = \frac{1}{\Gamma(\rho+1)}\int_0^\infty \frac{1}{A^2}\cdot\frac{u^\rho du}{\left(\frac{1}{A}e^u+1\right)^2}$$

$$= -\frac{1}{\Gamma(\rho+1)}\int_0^\infty \frac{1}{A}u^\rho\cdot\frac{\partial}{\partial u}\left(\frac{1}{\frac{1}{A}e^u+1}\right)du$$

$$= \frac{1}{A}U_{\rho-1}. \quad . \quad . \quad . \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Using relation (8), we get for

$$\delta S = \Sigma\left\{k\left[\frac{5}{2}\frac{U_{3/2}}{U_{1/2}} - \log A\right] + \frac{nk}{A}\left[\frac{3}{2} - \frac{5}{2}\frac{U_{3/2}\,.\,U_{-1/2}}{U^2_{1/2}}\right]\frac{\partial A}{\partial n}\right\}\delta n. \quad . \quad . \quad (9)$$

If Q is the heat absorbed due to the ionization of δn atoms,

$$\delta S = \frac{Q}{T} = \frac{-\chi\delta n + \Sigma\frac{\partial E}{\partial n}\,.\,\delta n}{T}. \quad . \quad . \quad . \quad (10)$$

Now E is given in the new statistics by the expression

$$E = \frac{3}{2}\frac{VGkT}{h^3}\,.\,(2\pi mkT)^{3/2}\,.\,U_{3/2}. \quad . \quad . \quad . \quad (11)$$

Then

$$\frac{\partial E}{\partial n} = \frac{3}{2}\frac{VGkT}{h^3}\,.\,(2\pi mkT)^{3/2}\frac{1}{A}U_{1/2}\,.\,\frac{\partial A}{\partial n}. \quad . \quad . \quad (12)$$

Hence we get, using relation (12),

$$\delta S = \frac{\chi}{T}\delta n + \Sigma \frac{3}{2} V k \beta_\nu . \frac{1}{A} U_{1/2} \frac{\partial A}{\partial n} \delta n, \quad . \quad . \quad (10\,a)$$

where we put

$$\beta_\nu = \frac{(2\pi m_\nu k T)^{3/2}}{h^3} . G. \quad . \quad . \quad . \quad . \quad (10\,b)$$

Equating (9) and (10 *a*), we get the general relation applicable to the new as well as the classical statistics,

$$\Sigma k \left[\frac{5}{2}\frac{U_{3/2}}{U_{1/2}} - \log A\right] + \Sigma \frac{nk}{A}\left[\frac{3}{2} - \frac{5}{2}\cdot\frac{U_{3/2} . U_{-1/2}}{U^2_{1/2}}\right]\frac{\partial A}{\partial n}$$
$$= -\frac{\chi}{T} + \frac{3}{2}\Sigma V k\beta . \frac{1}{A}\frac{\partial A}{\partial n} . U_{1/2}. \quad . \quad (13)$$

(3) *The Classical Ionization Formula.*

With the help of relation (13) we will arrive at the classical result (1) already quoted. The classical statistics correspond to a non-degenerate system, where in the distribution function $A << 1$.

In that case

$$U_\rho = U_{\rho-1} = \ldots . = U_{3/2} = U_{1/2} = U_{-1/2} = A \quad . \quad (14)$$

and
$$A = \frac{nh^3}{VG} \cdot (2\pi m k T)^{-3/2}$$

$$= \frac{n}{V\beta} \cdot \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Substituting relations (14) and (15) in (13), we get

$$-k\Sigma^\nu \log A_\nu = -\frac{\chi}{T}, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$-\log\frac{n}{V\beta} + \log\frac{n_+}{V\beta_+} + \log\frac{n_-}{V\beta} = -\frac{\chi^*}{kT}. \quad (16\,a)$$

Substituting values for β, we get at once the classical dissociation formula

$$\frac{n_+ n_-}{n} = \frac{V}{h^3} \cdot (2\pi m k T)^{3/2} . G e^{-\chi/kT}, \quad . \quad . \quad . \quad (17)$$

which is identical with equation (1). In the application of

* The positive sign is given to $\log n_+$ and $\log n_-$ because we have taken δn as an increase in the neutral atoms, and therefore δn_+ and δn_- are negative. Also

$$\delta n = \delta n_+ = \delta n_- .$$

this formula G can be omitted (G = 2, corresponding to spin statistical weight).

(4) *A Completely Degenerate System.*

For this we start again with the fundamental equation (13). In this case the values for U_ρ and A are completely different, and, as it will be seen, this complicates the ionization formula. Now

$$U_\rho = \frac{{u_0}^{\rho+1}}{\Gamma(\rho+2)} \times \left[1+2\left\{\frac{(\rho+1)\rho.c_2}{{u_0}^2}+\frac{(\rho+1)\rho.(\rho-1)(\rho-2)}{u_0^4}c_4,\ldots\right\}\right], \quad (18)$$

where

$$u_0 = \log A, \quad (18a)$$

and

$$c_\nu = 1-\frac{1}{2^\nu}+\frac{1}{3^\nu}-\frac{1}{4^\nu}, \quad (19)$$

giving in the special case for $\nu = 2$ the value $\pi^2/12$. The value for A in the case of degeneracy is also given by

$$\log A_\nu = \frac{h^2}{2m_\nu kT}\left(\frac{3n_\nu}{4\pi G}\right)^{2/3} \quad (20)$$

$$= \frac{\alpha n^{2/3}}{mkT}. \quad (21)$$

Equation (13) modifies by the substitution of the values for $U_{3/2}$, $U_{1/2}$, and $U_{-1/2}$ in the following manner:

$$\Sigma\frac{k\pi^2}{2}(\log A_\nu)^{-1}-\Sigma\frac{mk\pi^2}{2}(\log A_\nu)^{-2}.\frac{1}{A_\nu}\frac{\partial A_\nu}{\partial n} = -\frac{\chi}{T}+\frac{3}{2}\Sigma V k\beta_\nu.\frac{4}{3}\sqrt{\pi}(\log A_\nu)^{3/2} \times\left[1+\frac{\pi^2}{8(\log A_\nu)^2}\cdots\right]\frac{1}{A_\nu}\frac{\partial A_\nu}{\partial n}. \quad (22)$$

Relation (21) gives

$$\frac{1}{A}\frac{\partial A}{\partial n} = \frac{2}{3}\frac{\alpha n^{-1/3}}{mkT}. \quad (23)$$

Making all the necessary simplifications, we get

$$\Sigma\frac{k\pi^2}{6}\frac{mkT}{\alpha n^{2/3}}-\Sigma\frac{2^{7/2}.\pi^2 V\alpha^{5/2}G}{3h^3 mT}.n^{2/3} -\Sigma\frac{\pi^4 V}{6h^3}.\frac{2^{3/2}k^2\alpha^{1/2}mTG}{n^{2/3}} = -\frac{\chi}{T}. \quad (24)$$

Taking the volume as unity, we get our final ionization formula,

$$\frac{k^2\pi^2 T}{6}\left[\frac{1}{\alpha}-\frac{G\pi^2 2^{3/2}\alpha^{1/2}}{h^3}\right]\left[m\left(\frac{1}{n^{2/3}}-\frac{1}{n_+^{2/3}}\right)-\frac{n_-}{n_-^{2/3}}\right]$$
$$-\frac{2^{7/2}\pi^2\alpha^{5/2}G}{3h^3 T}\left[\frac{1}{m}(n^{2/3}-n_+^{2/3})-\frac{n_-^{2/3}}{m_-}\right]=-\frac{\chi}{T}. \quad . \quad (25)$$

(5) *Ionization Formula for the Degeneracy of the Electrons only.*

As has been pointed out, a modification of the ionization formula will be useful only *if the electrons alone are treated as degenerate.* This can easily be done, if in summing up the entropy of the system in equation (5), we use for the atoms the classical values for $U_{3/2}$, $U_{1/2}$, $U_{-1/2}$, and A, and the corresponding values in the case for degeneracy for the electrons. Using relations (16) and (24), we get our ionization formula, taking into consideration the degeneracy of the electrons alone :

$$k\log\frac{n_+}{n}-\frac{k^2\pi^2 T}{6}\left[\frac{1}{\alpha}-\frac{G\pi^2 2^{3/2}\alpha^{1/2}}{h^3}\right]m_- n_-^{-2/3}$$
$$+\frac{2^{7/2}\pi^2\alpha^{5/2}}{3h^3 T}\cdot\frac{n_-^{2/3}}{m_-}=-\frac{\chi}{T}. \quad . \quad . \quad (26)$$

Equation (26) can be put in the more convenient form

$$\log\frac{n_+}{n}=-\frac{\chi}{kT}+\Phi n^{-2/3}T-\Theta n^{2/3}T^{-1}, \quad . \quad . \quad (27)$$

where Φ and Θ have the obvious equivalents, which can be seen by comparison with equation (26). Finally, we get

$$\frac{n_+}{n}=e^{\Phi n^{-2/3}T-\Theta n^{2/3}T^{-1}-\chi/kT}. \quad . \quad . \quad . \quad (28)$$

(6) *Numerical Calculations and Discussion with Reference to the Theory of Dwarf Stars.*

Formulæ (17) and (28) are the classical and the modified ionization formulæ connecting the ionization potential, electron-density, and the temperature. If x is the degree of ionization, we get, on substituting the known numerical values for Φ and Θ, the final numerical formula in the two cases as

$$\log\frac{1-x}{x^2}=\frac{\chi}{kT}+\frac{3{\cdot}34\times10^{10}\times T}{n^{2/3}}+\frac{2{\cdot}1\times n^{2/3}}{10^{10}\times T}, \quad . \quad (29)$$

and

$$\frac{1-x}{x^2}=\frac{n\times4{\cdot}1\times10^{-16}}{T^{3/2}}e^{\chi/kT}. \quad . \quad . \quad . \quad (30)$$

The condition of degeneracy gives that the electrons are degenerate at a temperature of 10^9 Å. when the density is 10^{30}. Now we will calculate the degree of dissociation for various temperatures on both the formulæ for an ionization potential corresponding to $\lambda = 1{\cdot}22$ Å. The results are tabulated below :—

Electron density $= 10^{30}$ per c.c., $\lambda = 1{\cdot}22$ Å.

Temperature.	x by the classical formula	x by the modified formula (29).
A.	%	%
10^9	6·4	0
2×10^9	17	0
3×10^9	28	0
4×10^9	38	0
5×10^9	45	0

It is very interesting to note that our formula, including the degeneracy of the electrons, gives uniformly 0 per cent. ionization, while the classical formula gives varying degrees. As a matter of fact, it gives 0 per cent. up to a temperature above which the application of the formula itself becomes invalid, due to the Sommerfeld degeneracy condition not being satisfied. The degree of dissociation given by the classical formula has no significance, because the conditions on which the formula has been derived do not hold good at such high pressures.

Our theory predicting zero per cent. ionization goes against the fundamental assumption of Fowler, Stoner, and others in the theory of dwarf stars, where the first postulate is to take for granted the complete ionization in it. The temperature and density we have taken for our calculation of the table roughly correspond to the centre of the Companion of Sirius, and there our formula predicts no ionization.

If that is the case, one may be tempted to question the validity of the fundamental assumption in the theory of dwarf stars. But for all purposes of calculation we can take the electrons as free, if we define a free electron as one which is not under the influence of *one and the same* nucleus for any finite time. Then obviously the electrons at that pressure are free electrons. The "bound electrons" are changing partners continually, and go about knocking, thus really

travelling the whole of the phase-space, just what a "free electron" is expected to do. But, none the less, the statement that *every electron* is under the influence of one or other of the nuclei at any arbitrarily chosen time remains valid. In other words, the ionization is zero, a result predicted by our theory. Thus the paradoxical statement results that the electrons, though all of them bound, are free.

(7) *Summary.*

In this paper the ionization formula is considered afresh on the Fermi-Dirac statistics. The modified formula where the degeneracy of the electrons is considered gives, under the conditions existing in the interior of a dwarf star, a degree of ionization = 0 per cent. This result is discussed with reference to the theory of dwarf stars, where complete ionization is usually assumed.

The Presidency College,
Madras, India,
23rd May, 1929.

XXVIII. *The Raman-Effect in the Proximity of the Critical Point.* *By* S. L. Ziemecki, *Ph.D., M.Sc., Lecturer in Physics, High School of Agriculture, Warsaw, and* K. Narkiewicz-Jodko *.

1. *The Aim of this Work.*

EXPERIMENTING with liquids, we find that some of them give easily strong sharp Raman lines, whilst others require long exposures to obtain Raman spectra consisting of faint badly defined bands. We give below some figures concerning the relative scattering powers of liquids †.

Relative Scattering Powers of Liquids.

Ethyl-ether	1·00
Water	0·21
Acetic Acid	1·19
Benzene	3·15
Paraxylene	4·61
Carbon disulphide	13·0

* Communicated by the Authors. Read at the meeting of the Polish Physical Society on October 21st, 1929.

† See J. Cabannes, 'La diffusion moléculaire de la lumière,' Paris, 1929, p. 188.

Certainly it is remarkable that water, which has the smallest scattering power, is also one of the liquids giving the faintest Raman bands, while benzene and its derivatives and carbon disulphide, having great scattering power, give strong Raman spectra. This suggestion has a purely qualitative character, but shows that it would not be quite unreasonable to look for a certain connexion between the Raman effect and classical light scattering.

If we admit that such a connexion exists, we would expect a considerable increase in the intensity of the Raman effect in the proximity of the critical point, where the intensity of classical scattering is increasing hundreds or even thousands of times, causing the phenomenon of critical opalescence. This question was recognized by Raman * himself to be of fundamental importance. Raman experimented with CO_2 and further with a mixture of carbon disulphide and methyl-alcohol. In both those cases he found *an increase* of the new radiation in the proximity of the critical point. L. A. Ramdas † repeated the experiment with CO_2, and also observed an increased intensity of the effect. The question was taken up by A. Bogros and Y. Rocard ‡, and also by W. H. Martin §. These authors, working with a critical mixture of phenol and water (64 per cent. of water), obtained *absolutely negative* results ; they observed no Raman spectra even after an exposure of *three hours*.

So far the problem remained unsolved. Raman himself considered his results as preliminary ; he admitted that deeper research would be necessary, because he limited himself to the method of complementary light filters, and did not obtain photographs of the spectra. On the other hand, the experiments of Martin and Bogros-Rocard seemed indecisive : phenol mixtures quickly become coloured, and one could think the negative results were due to the absorption of the new radiation. Whatever it may have been, the above authors should have obtained in their photographs at least the diffused Raman bands of water, because the Raman effect is strictly additive, and with pure water one easily obtains the Raman spectrum

* Indian Journ. of Physics, ii. pt. 3, p. 387 (1928).
† Ind. Journ. of Physics, iii. pt. 1, p. 131 (1928).
‡ *Comptes Rendus*, clxxxvi. p. 1712 (1928).
§ 'Nature,' October 6, 1928, p. 506.

with an exposure of $1\frac{1}{2}$ hours. To elucidate the problem, the authors undertook a new line of investigation. The results obtained seem to give a decisive answer to the question, and explain also why the former scientists obtained contradictory results.

2. *Experimental Arrangement.*

We used in our experiments isobutyric acid, which with water gives a mixture having the critical temperature at about 24° C.; the mixture contains at the critical point 36·3 per cent. (weight) parts of acid. The isobutyric acid was obtained from Kahlbaum. The preparation, having been in continual use for many months, remained absolutely colourless and possessed the same optical properties as a new sample manufactured by Kahlbaum-Schering.

For a source of light we used a mercury-vapour lamp of a special type*. The arc was formed in quartz tube of 2 cm. diameter, its length was 20 cm. The anode was cooled by a quick water stream. Thus heavy currents could be used, amounting to 10 amp. in our experiments. The lamp was continuously connected with a mercury diffusion pump.

The light of the lamp was concentrated at the axis of a cylindrical vessel R (see figs. 1–3) by means of a condenser of 16 cm. in diameter. The vessel had flat ends; the end turning to the collimator of the spectrograph was made from a plane-parallel plate fused in a glass tube. Sibor glass was used. The cylindrical part of the vessel R was half-silvered; this silver mirror increased the intensity of the light. To keep a steady temperature the vessel R was immersed in a metal tank filled with water of 14 litres capacity. The water was made to circulate by means of the stirrers S, moved by a motor. The heating of the water-bath was effected by means of the chromium-nickel spiral P. The temperature was easily kept constant to 0·1° C. A Leiss glass-spectrograph, model A, was used. In the green-blue part of the spectrum we had 50 Å for 1 mm. The error of the wave-length determinations should not exceed 2–3 Å units. We used Ilford Iso-Zenith plates

* The model of this lamp was constructed by Prof. S. Pienkowski at the Institute for Experimental Physics at the Warsaw University. We are glad to express here our best thanks to Prof. Pienkowski for his kind permission to adapt his model.

Fig. 1.

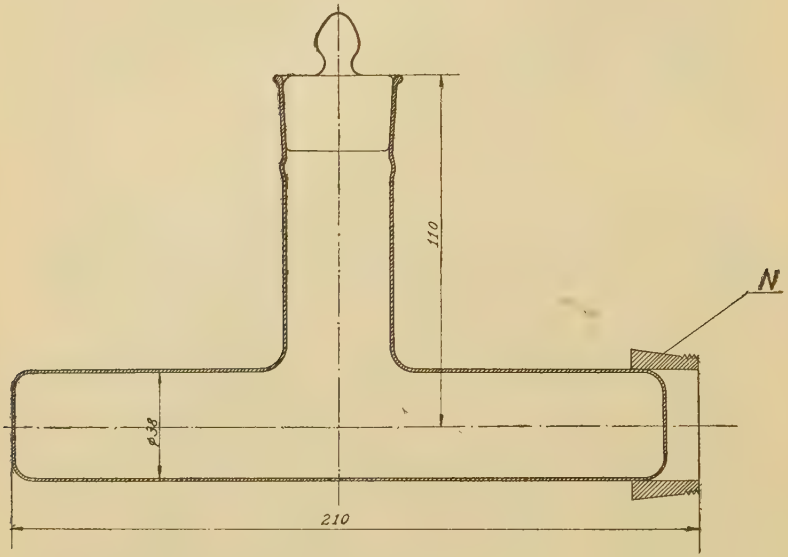
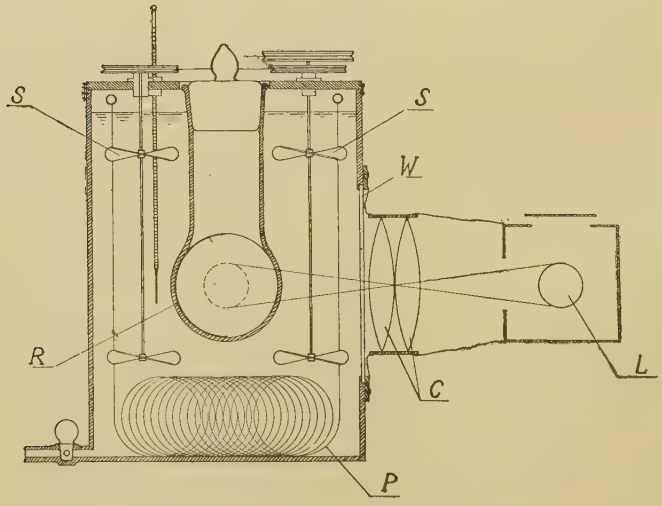


Fig. 2.



Explanation of lettering on figs. 1–3.

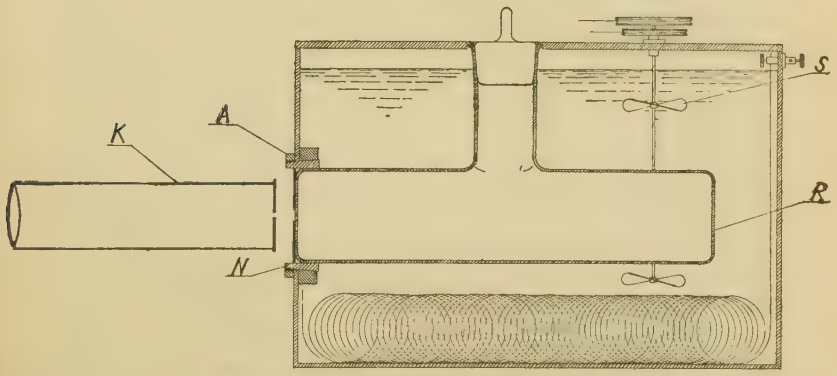
N is a brass cone which fits into the cone A (fig. 3). S denotes the vanes of the stirrers. R denotes the glass vessel containing the liquid examined. W is a glass window. L the mercury lamp, whose image is projected by the condenser lenses C. P is the heating spiral. K the collimator of the spectrograph.

(H. & D. 700). For the photographs taken during the opalescence it was found advantageous to apply anti-halo plates, the principal mercury lines being then strongly superexposed. We chose Gevaert's SSS (Super Sensima Special) plates, which gave very satisfactory photographs of the spectra.

3. *Results.*

Our experiments were conducted in the following way. We photographed the Raman spectrum some degrees over the critical point, when no distinct opalescence was yet seen. Then, allowing the liquid to cool down, we repeated the experiment, applying the same time of exposure, while

Fig. 3.



the opalescence was increasing in strength. We aimed not to reach the point at which the liquid would look like diluted milk, because then the absorption of the radiation would have been too strong. The first photographs taken soon showed that the opalescence was accompanied by the appearance of many new lines which were not visible at higher temperatures. Relatively faint characteristic Raman lines of isobutyric acid also appeared, but these were more diffused than the same lines obtained with pure acid. The attempts to obtain more intense phenomena by prolonging the time of exposure did not bring expected results. The lines disappeared altogether, and a reason for their diappearance was easily found. The spectrum of the mercury-arc lamp has a weak continuous background. In ordinary conditions this continuous spectrum does not

appear on photographs, because the scattered light amounts to only a small fraction of the light incident. At the critical opalescence that fraction grows considerably, and the background becomes relatively important. In cases of long exposure the continuous spectrum totally masks the weak Raman lines. In view of the above the following method of observation was adopted. We shortened the length of exposure to the length of time necessary to obtain on the plate the strongest broad Raman-lines of isobutyric acid only — 4605 Å and 4661 Å. Fifteen minutes' exposure was found sufficient for this purpose. Under these conditions the continuous background was relatively harmless. Lowering the temperature and approaching the critical point, the time of the exposure remaining unchanged, we took a series of spectrum photographs. These photographs showed very clearly that the intensity of the Raman lines changed but very little, whereas the intensity of the classical scattering increased manyfold. It is true that at intense opalescence the Raman lines appeared somewhat stronger than without opalescence, but at the same time the continuous background was darker. For an approximate estimation of the strengthening of the lines we diminished the time of exposure ; the temperature and the degree of opalescence remained the same. It was thus found that the strengthening of the Raman lines at maximum opalescence in our experiments did not exceed 30–40 per cent. However, we do not think that it was a real increase in the intensity of the lines, but consider it rather to be the effect of the simultaneous action on the plate of both the Raman light and also the continuous spectrum of increased intensity. At the same time the intensity of the scattered light increased approximately about 500 per cent. This effect could not be appreciated with the principal mercury lines, because they appeared on the plates much overexposed, but it was perceptible with the weak ones like 4916 Å. It was said that there appeared at the opalescence many new lines, which we thought at the beginning to be new Raman lines. A more exhaustive study showed, however, that these new lines were but very feeble mercury lines, like 4826, 4882, 4960, 5121, 5365 Å. They do not appear in the spectra of scattered light when the temperature is kept some degrees above the critical point, their intensity being too weak to

impress the photographic plate. The opalescence increases their intensity in the scattered light. It was possible for us to obtain the same "new lines," using instead of the critical mixture a plate of milky glass put in the vessel R (see fig. 2), which was filled with distilled water. A three minutes' exposure was then sufficient to obtain on the plate the lines mentioned.

4. *Conclusion.*

From these experiments we infer that the Raman spectrum exists at the critical point, and its intensity remains unchanged by the critical opalescence.

The results of the investigations of Bogros-Rocard * and Martin† are easily explained. The fact that these scientists did not obtain Raman spectra in their photographs resulted from the continuous background of the mercury spectrum, which, becoming predominant at the opalescence, masks totally the weak Raman lines. Apart from this, in the case of phenol the liquid becomes red-coloured and the absorption has a harmful effect upon the intensity of the Raman light.

The positive effect of Raman and Ramdas is accounted for if we consider that it is impossible to obtain ideal light-filters. Their blue filter transmitted also a certain percentage of green and yellow light; the amount of light transmitted was too small to be observed in ordinary conditions of light scattering. At the critical opalescence the scattering increased considerably, and the light transmitted appeared suddenly, thus giving the impression of strengthened Raman radiation.

It follows from our work that there exists a fundamental difference between Raman effect and scattered radiation, and that, on an *experimental basis*, we may call the new radiation incoherent, whereas the classical scattering is a coherent one.

A part of the expense for this study was covered by the Mianowski Foundation.

Warsaw, Physical Laboratory of
the State High Technical School,
founded by H. Wawelberg and S. Rotwand,
November 3rd, 1929.

* *Loc. cit.* † *Loc. cit.*

XXIX. *The Theory of the Pianoforte String struck by a Hard Hammer.*—Part I. *By* Dr. KULESH CHANDRA KAR *and* MOHINIMOHON GHOSH *.

Introduction.

THE theoretical inquiries into the problem of the pianoforte string are generally divided into the two following classes :—(1) those developed by Helmholtz †, Delemer ‡, and Lamb § ; (2) those advanced by Kaufmann ‖, Love ¶, and Das **. It is generally thought that the two classes of theories are fundamentally different, the first class being based on some assumed law of pressure between the hammer and the string, the second class giving a rigorous functional solution of the problem. It is proposed in this paper to develop a new theory of the pianoforte string, extending Rayleigh's theory of loaded string †† in the light of the theory of motion of dynamical systems under intermittent impulse as developed by one of the writers ‡‡ on different occasions, and to discuss the present theory in relation to the other theories of class (2).

Before entering into discussions about the present theory and its relation to the other existing theories it seems advisable to state briefly the results of Kaufmann's investigation §§ on the subject. According to Kaufmann, the differential equation of motion of the hammer when it strikes very near the end of the string is given by

$$m\frac{d^2y}{dt^2}+\frac{T_1}{c}\cdot\frac{dy}{dt}+\frac{T_1}{a}y=0, \quad . \quad . \quad . \quad . \quad (1)$$

where y is the displacement of the hammer, t the time, m the effective mass of the hammer, T_1 the tension of the string, c the velocity of the wave along the string, and a

* Communicated by the Authors.

† Helmholtz, 'Sensations of Tone,' Translated by Ellis.

‡ Delemer, *Ann. Soc. Sci. de Bruxelles*, pts. 3-4, pp. 219-314.

§ Lamb, 'The Dynamical Theory of Sound,' p. 74.

‖ Kaufmann, *Ann. d. Phys.* liv. p. 675.

¶ Love, 'The Theory of Elasticity,' 4th Edition, Art. 281.

** Das, Proc. Ind. Assoc. etc. vii. p. 13 (1921), *et seq.*

†† Rayleigh, 'Theory of Sound,' i. p. 204.

‡‡ K. C. Kar, *Phys. Zeit.* xxiv. p. 63 (1923): Phys. Rev. xxi. p. 695 (1923); Phil. Mag. (to be published) Kar and Ghosh ; *Phys. Zeit.* xxix. p. 143 (1928) ; M. Ghosh, *Phys. Zeit.* (to be published).

§§ Kaufmann, *loc. cit.* Notations used in this paper are generally taken from Rayleigh's 'Theory of Sound.'

the shorter segment into which the string is divided by the hammer. It may be noted that the above equation holds during the time $0 < ct < 2a$. Now on solving the above equation one gets

$$y = \frac{v_0}{\sqrt{\frac{T_1}{ma} - \frac{T_1^2}{4m^2c^2}}} \cdot e^{-\frac{T_1}{2mc}} \sin\left\{ t\sqrt{\frac{T_1}{ma} - \frac{T_1^2}{4m^2c^2}} \right\}, \quad (2\,a)$$

$$0 < ct < 2a,$$

or

$$y = \frac{v_0}{c\sqrt{\frac{\rho}{ma} - \frac{\rho^2}{4m^2}}} \cdot e^{-\frac{\rho c}{2m} \cdot t} \sin ct \left\{ \frac{\rho}{ma} - \frac{\rho^2}{4m^2} \right\}^{\frac{1}{2}}, \quad . \quad . \quad (2\,b)$$

or again,

$$y = \frac{v_0\theta}{\sqrt{\frac{m'}{m}\left(\frac{4l}{a} - \frac{m'}{m}\right)}} \cdot e^{-\frac{m'}{m} \cdot \frac{t}{\theta}} \sin\left\{ \frac{t}{\theta}\sqrt{\frac{m'}{m}\left(\frac{4l}{a} - \frac{m'}{m}\right)} \right\}, \quad (2\,c)$$

where ρ is the linear density of the string, θ the period of vibration, v_0 the initial velocity of the struck point, l and m is length and mass respectively. Thus from (2 *c*) the duration of contact is given by

$$\frac{\phi}{\theta} = \frac{\pi}{\sqrt{\frac{m'}{m}\left(\frac{4l}{a} - \frac{m'}{m}\right)}}. \quad . \quad . \quad . \quad . \quad (3)$$

The next case considered by Kaufmann is that of a hammer striking at the centre of the string. In this case the differential equation of motion of the hammer is given by

$$m\frac{d^2y}{dt^2} = -2T_1 \cdot \frac{dy}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where m and T_1 are symbols already explained and x is taken along the string. The solution of the above equation in the first epoch, *i. e.*, $0 < ct < l$, is

$$y = \frac{v_0 mc}{2T_1}\left(1 - e^{-\frac{2T_1}{mc}t}\right), \quad . \quad . \quad . \quad . \quad (5\,a)$$

or,

$$y = \frac{mv_0}{2\rho c}\left(1 - e^{-\frac{2\rho c}{m}t}\right), \quad . \quad . \quad . \quad . \quad (5\,b)$$

which may also be written as

$$y = \frac{v_0 \theta m}{4m'}\left(1 - e^{-\frac{4m'}{m}\cdot\frac{t}{\theta}}\right), \quad . \quad . \quad . \quad . \quad (5\,c)$$

where the symbols have the same meaning as in (2). In the second epoch the solution of equation (4) becomes

$$y = \frac{v_0}{c}\left\{e^{-\frac{2T_1}{mc}\cdot t}\left[2e^{\frac{4T_1 a}{mc}}\left(ct - 2a + \frac{mc^2}{2T_1}\right) - \frac{mc^2}{2T_1}\right] - \frac{mc^2}{2T_1}\right\}, \quad (6\,a)$$

or,

$$y = \frac{mv_0}{2\rho c}\left[e^{-\frac{2\rho c}{m}t}\cdot\frac{2\rho}{m}\left\{2e^{\frac{4a\rho}{m}}\left(ct - 2a + \frac{m}{2\rho}\right) - \frac{m}{2\rho}\right\} - 1\right], \quad (6\,b)$$

or, in the alternative form,

$$y = v_0\theta\left\{e^{-\frac{4m'}{m}\cdot\frac{t}{\theta}}\left[2e^{\frac{2m'}{m}}\left(\frac{t}{\theta} - \frac{1}{2} + \frac{m}{4m'}\right) - \frac{m}{4m'}\right] - \frac{m}{4m'}\right\}. \quad (6\,c)$$

Now, on equating $\frac{d^2y}{dt^2}$ from (6) to zero, the duration of contact of the hammer, if it leaves the string in the second epoch, can be easily obtained. It is, according to Kaufmann,

$$\frac{\phi}{\theta} = \frac{1}{2} + \frac{1}{4}\cdot\frac{m}{m'}\left(1 + \frac{1}{2}e^{-\frac{2m'}{m}}\right). \quad . \quad . \quad . \quad (7)$$

One other case discussed by Kaufmann in his paper referred to above is that of an infinite string struck at its centre. The solution for this case is exactly the same as equation (5) for the first epoch for a finite string, for reasons which are quite obvious. It may be noted before concluding this section that Kaufmann's method for a finite string struck at its centre can be easily extended to higher epochs. The displacement in the third epoch may conveniently be given below,

$$y = \frac{mv_0}{2\rho c}\left[1 - e^{-\frac{2\rho c}{m}t} + 2e^{-\frac{2\rho}{m}(ct-2a)}\cdot\frac{2\rho}{m}\left(ct - 2a + \frac{m}{2\rho}\right)\right.$$
$$\left. - 2e^{-\frac{2\rho}{m}(ct-4a)}\cdot\frac{2\rho}{m}\left\{(ct-4a) + \frac{2\rho}{m}(ct-4a)^2 + \frac{m}{2\rho}\right\}\right]. \quad (8)$$

The above is a brief summary of Kaufmann's theory of the pianoforte string.

In the following section A we shall develop the new theory of the pianoforte string, and shall show that it gives all the results given above in a very simple way. In section B will be discussed the relations of the theories of

Love and Das with that of Kaufmann, while in section C will be given discussions about the different expressions for the duration of contact obtained from the different theories.

Section A.

In all the existing theories of the pianoforte string the effect of damping has been neglected. In the present paper we shall also neglect the damping effect, which will, however, be considered in a later communication. Now, it has been shown by one of the writers * that if the vibration of the pianoforte string during the time the hammer is in contact is taken as that of a loaded string, the load being considered as a part of the string during this time, then the equation of motion of the load or the loaded point of the string is given by

$$y=\frac{2v_0}{c\lambda}\sin c\lambda t\,.\,\frac{1}{\frac{\rho}{m}\left(\frac{a}{\sin^2\lambda a}+\frac{b}{\sin^2\lambda b}\right)+1}, \quad . \quad (9)$$

where v_0 is the velocity of the hammer just before contact, c the velocity of the wave along the string, ρ the linear density of the string, m the mass of the hammer, a and b are the segments into which the string is divided by the hammer, and λ is given by the frequency equation

$$m\lambda\sin\lambda a\sin\lambda b=\rho\sin\lambda l, \quad . \quad . \quad . \quad . \quad (10)$$

l being the total length of the string.

Near the End.

Now, if the hammer strikes the string very near its end, *i. e.*, if $a<<b$ and therefore $b=l$ approximately, we have then from equation (10), by taking $\sin\lambda a=\lambda a$ and $\sin\lambda b=\sin\lambda l$,

$$\lambda=\sqrt{\frac{\rho}{ma}}=\frac{1}{c}\sqrt{\frac{T_1}{ma}}. \quad . \quad . \quad . \quad (11)$$

Thus equation (9) becomes

$$y=\frac{2v_0}{\sqrt{\frac{T_1}{ma}}}.\sin\sqrt{\frac{T_1}{ma}}.t\times P, \quad . \quad . \quad . \quad (12)$$

* K. C. Kar, Phil. Mag. p. 278 (August 1928). The value of y is not taken here as a series. This is what I pointed out in my paper referred to above.—K. C. K.

where $$P = \frac{1}{\frac{\rho}{m}\left(\frac{a}{\sin^2 \lambda a} + \frac{b}{\sin^2 \lambda b}\right) + 1} \quad . \quad . \quad . \quad . \quad (13)$$

Now, from equation (11) we have λ is very small. If, however, it is so small that $\sin \lambda a = \lambda a$ and $\sin \lambda b = \lambda b$, then ρ can be easily evaluated from (13). Thus we have

$$P = \frac{1}{\frac{1}{\lambda^2} \cdot \frac{\rho}{m}\left(\frac{1}{a} + \frac{1}{b}\right) + 1} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Or, substituting the value of λ from (11), and remembering that $b = l$ approximately, we have from (14) $P = \frac{1}{2}$. Thus equation (12) reduces to

$$y = \frac{v_0}{\sqrt{\frac{T_1}{ma}}} \cdot \sin \sqrt{\frac{T_1}{ma}} \cdot t. \quad . \quad . \quad . \quad (15)$$

It is not difficult to see that up to a first approximation, the above equation is the same as Kaufmann's equation (2). Let us now proceed to find the value of y up to a higher approximation *.

Now, using the relation $l = a + b$, we have from the frequency equation (10)

$$\lambda\left(\lambda a - \frac{\lambda^3 a^3}{6}\right) = \frac{\rho}{m}\left[\cos \lambda a + \frac{\sin \lambda a \cos \lambda b}{\sin \lambda b}\right], \quad . \quad (16)$$

neglecting higher terms.

Now, as λ is very small (shown before), $\sin \lambda a$, $\sin \lambda b$, and $\cos \lambda b$ may be taken as λa, λb, and 1 respectively. Thus equation (16) reduces to

$$\lambda\left(\lambda a - \frac{\lambda^3 a^3}{6}\right) = \frac{\rho}{m}\left(\cos \lambda a - \frac{a}{b}\right). \quad . \quad . \quad . \quad (17)$$

Now, $\frac{a}{b}$ being very small compared with $\cos \lambda a$, which is very nearly equal to 1, we have from (17)

$$\lambda^2 a\left(1 - \frac{\lambda^2 a^2}{6}\right) = \frac{\rho}{m}\left(1 - \frac{\lambda^2 a^2}{2}\right). \quad . \quad . \quad . \quad (18)$$

* An alternative proof will be given in Appendix (A).

Or we have on transformation

$$\lambda^2 a(1+\tfrac{1}{3}\lambda^2 a^2)=\frac{\rho}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Solving the above quadratic equation in λ^2, we get

$$\lambda=\sqrt{\frac{\rho}{ma}-\frac{1}{3}\cdot\frac{\rho^2}{m^2}}, \quad . \quad . \quad . \quad . \quad (20)$$

up to a second approximation. Thus instead of equation (15) we have

$$y=\frac{v_0}{c\sqrt{\frac{\rho}{ma}-\frac{1}{3}\frac{\rho^2}{m^2}}}\cdot\sin\left\{ct\sqrt{\frac{\rho}{ma}-\frac{1}{3}\cdot\frac{\rho^2}{m^2}}\right\}. \quad . \quad (21)$$

If we, however, suppose the term $\frac{1}{3}\cdot\frac{\rho^2 c^2}{m^2}$ under the square root in the above equation to be due to some term in the differential equation of motion corresponding to the damping term (as in Kaufmann), then the equation (21) may be written as

$$y=\frac{v_0}{c\sqrt{\frac{\rho}{ma}-\frac{\rho^2}{3m^2}}}\cdot e^{-\frac{1\cdot15\cdot\rho c}{2m}t}\cdot\sin\left\{ct\sqrt{\frac{\rho}{ma}-\frac{1}{3}\cdot\frac{\rho^2}{m^2}}\right\}, \quad . \quad (22)$$

which very closely agrees with Kaufmann's equation (2 *b*). The small difference may be explained as being due to the fact that Kaufmann has taken the smaller segment to be a rigid rod, which, however, is not so actually.

At Midpoint (First Epoch).

In considering the general case of equation of motion of a loaded string, Lord Rayleigh has considered the force to be of the form e^{ipt}. It can be easily seen, however, that during the first epoch, *i.e.*, before the time the reflected wave comes back to the load, the force will be of the form e^{-qt}. Thus the equation of motion of the load becomes (*vide* Rayleigh, *loc. cit.*)

$$T_1\cdot\Delta\left(\frac{dy}{dx}\right)+Fe^{-qt}=0, \quad . \quad . \quad . \quad . \quad (23)$$

where T_1 is the tension and c the velocity of the wave along the string. Thus from (23) the displacement of the load at any time is given by

$$y_1=\gamma e^{-qt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where γ is a constant. Now at $t=0$ we have $y_0=\gamma$. If we, however, suppose the displacement of the load to be zero at $t=0$, that is to say, if we measure the displacement from $y_1=y_0$, the above equation (24) becomes

$$y=\gamma(e^{-qt}-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Now, the velocity of the load just before contact with the string is $\left(\frac{dy}{dt}\right)_{t=0}=v_0$, and hence we get from (25) $\gamma=-\frac{v_0}{q}$, and thus we have

$$y=\frac{v}{q}(1-e^{-qt}). \quad . \quad . \quad . \quad . \quad . \quad (26)$$

It may be noted here that as we have taken the force in the first epoch to be of the form e^{-qt}, the frequency equation (10) will be (*vide* Rayleigh, Theory of Sound, *loc. cit.*)

$$m\lambda \sin \lambda a \sin \lambda b=-\rho \sin \lambda l. \quad . \quad . \quad . \quad (27)$$

Now at the centre $a=b=\frac{l}{2}$, and thus we have

$$\lambda=-\frac{2\rho}{m}\cdot\frac{\cos \lambda a}{\sin \lambda a}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Or, substituting the exponential values of $\cos \lambda a$ and $\sin \lambda a$, we get

$$\lambda=-i\cdot\frac{2\rho}{m}\cdot\frac{e^{i\lambda a}+e^{-i\lambda a}}{e^{i\lambda a}-e^{-i\lambda a}}. \quad . \quad . \quad . \quad . \quad (29)$$

Or again, by putting $p=\lambda c$ (*vide* Rayleigh, *loc. cit.*), we have

$$p=-i\frac{2\rho c}{m}\cdot\frac{e^{ipa/c}+e^{-ipa/c}}{e^{ipa/c}-e^{-ipa/c}}. \quad . \quad . \quad . \quad . \quad (30)$$

It has been shown above that in the present case $p=iq$. Thus we have from (30)

$$q=-\frac{2\rho c}{m}\cdot\frac{e^{-qa/c}+e^{qa/c}}{e^{-qa/c}-e^{qa/c}}. \quad . \quad . \quad . \quad . \quad (31)$$

We have already pointed out in the introduction that the solution for a finite string during the first epoch will be exactly the same as that for an infinite string. Thus the value of q becomes on putting $l=\infty$, *i. e.*, $a=\infty$,

$$q=\frac{2\rho c}{m}=\frac{2T}{mc}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence equation (26) becomes

$$y = \frac{v_0 mc}{2T}\left(1 - e^{-\frac{2T}{mc}\cdot t}\right), \quad . \quad . \quad . \quad (33)$$

which is Kaufmann's equation (5 *a*) for the first epoch or for an infinite string. We might point out here that the differential equation corresponding to (33) during this time is (*vide* also Kaufmann, *loc. cit.*)

$$m \cdot \frac{d^2y}{dt^2} + \frac{2T}{c} \cdot \frac{dy}{dt} = 0. \quad . \quad . \quad . \quad . \quad (34)$$

Second and Higher Epochs.

In calculating the displacement of the hammer during the second epoch, Kaufmann has adopted the method known as the method of variation of integration constant or Saint Venant's method. Love has also used the same method of Saint Venant to find the solution during the second epoch for a rod struck at an end. It will be shown in section B that both the equations of Kaufmann and Love, though apparently of different form, are really the same. In the present section we shall, however, use Kar's theory of intermittent action which has found application * in different branches of Physics to get very easily the displacement of the hammer during the second or subsequent epoch.

We suppose, now, that during the time it is in contact with the string the hammer moves on, imparting and thus getting back impulses at regular intervals of time $= \frac{2a}{c}$, being the time taken by the wave to start from the hammer or the load and come back to it after reflexion from fixed ends. The magnitude of these sudden increases of impulsive force or pressure at the centre, as taken by Kaufmann and Das, are $2\rho vc$, $4\rho vc$, $4\rho vc$..... etc. at $t = 0$, $\frac{2a}{c}$, $\frac{4a}{c}$ .... etc. Thus the motion of the load or the hammer during any epoch is given by the differential equation (34) with the above conditions of impulsive pressure. Now it follows from the theory of intermittent action, that if a pressure $2\rho vc$ acts momentarily at $t = 0$, on a system given by the equation (34), the displacement at any time t is given by (33) †. Then again, if the momentary increases of

* *Loc. cit.* (Introduction).

† See Appendix B for proof.

pressure $2\rho vc$ and $4\rho vc$ take place at $t=0$ and $\frac{2a}{c}$, the displacement at any time during the second epoch, *i. e.*, between $\frac{2a}{c}$ and $\frac{4a}{c}$, is given by

$$y_{\frac{2a}{c}+\mathrm{T}} = \frac{mv_0}{2c\rho}\left[\left\{1-e^{-\frac{2c\rho}{m}\left(\frac{2a}{c}+\mathrm{T}\right)}\right\}-2\left(1-e^{-\frac{2c\rho}{m}\cdot\mathrm{T}}\right)\right]. \quad (35)$$

Or on simplifying we get

$$y_{\frac{2a}{c}+\mathrm{T}} = \frac{mv_0}{2c\rho}\left\{e^{-\frac{2c\rho}{m}\cdot\mathrm{T}}\left(2-e^{-\frac{4a\rho}{m}}\right)-1\right\}. \quad . \quad (36)$$

It may be noted here that T in the above equation is the time measured from the end of the first epoch. Thus, putting $t=\frac{2a}{c}+\mathrm{T}$ in Kaufmann's equation (6 *b*) we get, after some transformation,

$$y = \frac{mv_0}{2c\rho}\left[2e^{-\frac{2\rho c}{m}\cdot\mathrm{T}}\cdot\frac{2\rho}{m}\left(c\mathrm{T}+\frac{m}{2\rho}\right)-e^{-\frac{2\rho}{m}(c\mathrm{T}+2a)}-1\right]. \quad (37)$$

Now it can be easily seen that if in the above equation cT is neglected compared with $\frac{m}{2\rho}$, the equation (37) becomes identical with equation (36). We may also note in this connexion that cT will generally be negligible in comparison with $\frac{m}{2\rho}$. Thus there is very good agreement between the present theory and that of Kaufmann. The small difference between the expressions obtained from the two theories is due to the fact that in Saint Venant's method the pressure between the hammer and the string has a sudden increase at the end of each epoch, and then it exponentially decreases, whereas according to the present theory the sudden increase of pressure takes place momentarily at the end of every epoch. This point, it may be remarked here, will be thoroughly discussed in Part II.*

The displacement of the load or the hammer during the third epoch can be easily obtained as before. Now we have in the third epoch,

$$y_{\frac{4a}{c}+\mathrm{T}} = \frac{mv_0}{2\rho c}\left[\left\{1-e^{-\frac{2\rho c}{m}\left(\mathrm{T}+\frac{4a}{c}\right)}\right\}-2\left\{1-e^{-\frac{2\rho c}{m}\left(\mathrm{T}+\frac{2a}{c}\right)}\right\}\right.$$
$$\left.+2\left\{1-e^{-\frac{2\rho c}{m}\cdot\mathrm{T}}\right\}\right]. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

* See p. 321 infrà.

On simplifying, we get

$$y_{\frac{4a}{c}+\mathrm{T}} = \frac{mv_0}{2\rho c}\left[1-e^{-\frac{2\rho c}{m}.\mathrm{T}}\ 2-2e^{-\frac{4a\rho}{m}}+e^{-\frac{8a\rho}{m}}\right]. \quad . \quad (39)$$

Now, on substituting $t=\frac{4a}{c}+\mathrm{T}$, the corresponding equation (8) of Kaufmann reduces to

$$y_{\frac{4a}{c}+\mathrm{T}} = \frac{mv_0}{2\rho c}\left[1-e^{-\frac{2\rho}{m}(c\mathrm{T}+4a)}+\frac{4\rho}{m}e^{-\frac{2\rho}{m}(c\mathrm{T}+2a)}\right.$$
$$\times\left\{(c\mathrm{T}+2a)+\frac{m}{2\rho}\right\}-4e^{-\frac{2\rho c}{m}.\mathrm{T}}.\frac{\rho}{m}$$
$$\left.\times\left\{c\mathrm{T}+\frac{2\rho}{m}c^2\mathrm{T}^2+\frac{m}{2\rho}\right\}\right]. \quad . \quad . \quad . \quad . \quad (40)$$

If, as before, we assume $c\mathrm{T}$ as small compared with $\frac{m}{2\rho}$, the above equation (40) becomes exactly the same as (39) deduced from the theory of intermittent action. It may not be too much to claim at this stage that Kar's theory of intermittent action gives quite easily a value of the displacement of the hammer during any epoch which very closely approximates to the corresponding value obtained by a long process based on Saint Venant's method of variation of integration constant.

Section B.

Love's Theory.

As has been already pointed out, Love, in his book on Elasticity, has discussed the case of a rod struck by a load or hammer at one of its ends, the other end being fixed. The time during which the hammer is in contact with the rod is divided into a number of equal intervals, each being equal to $\frac{2a}{c}$, where a and c are respectively the length of the rod and the velocity of propagation of a wave along it. And the displacements for the first, the second, and the third epoch when expressed in the notations used in this paper, are :

first epoch,

$$y_{0<t<\frac{2a}{c}} = \frac{mv_0}{\rho c}\left(1-e^{-\frac{\rho c}{m}t}\right); \quad . \quad . \quad . \quad (41)$$

second epoch,

$$y_{\frac{2a}{c}<t<\frac{4a}{c}} = \frac{mv_0}{\rho c}\left[e^{-\frac{\rho c}{m}t}\cdot\frac{\rho}{m}\left\{2e^{-\frac{2\rho a}{m}}\left(ct-2a+\frac{m}{\rho}\right)-\frac{m}{\rho}\right\}-1\right]; \quad (42)$$

third epoch,

$$y_{\frac{4a}{c}<t<\frac{6a}{c}} = \frac{mv_0}{\rho c}\left[1-e^{-\frac{\rho c}{m}t}+2e^{-\frac{\rho}{m}(ct-2a)}\left\{1+\frac{\rho}{m}(ct-2a)\right\}\right.$$
$$\left.-2e^{-\frac{\rho}{m}(ct-4a)}\left\{1+\frac{\rho}{m}(ct-4a)+\frac{\rho^2}{m^2}(ct-4a)^2\right\}\right] \quad . \quad (43)$$

Now it is quite evident that, by putting $\frac{m}{2}$ for m in equations (41), (42), and (43), we get the corresponding equations (5 *b*), (6 *b*), and (8) of Kaufmann for the hammer struck at the centre of the string. This is not at all surprising in view of the fact that both Kaufmann and Love have used the same method of variation of integration constant.

P. Das's Theory.

In developing his theory of the pianoforte string P. Das * has assumed that at the end of every epoch two similar waves are set up, and immediately they are formed they start from the struck point in two opposite directions. Now, to evaluate the unknown function representing the two similar waves produced, Das falls back on the well-known method of variation of integration constant adopted before by Kaufmann and Love. So, we think, Das's theory is fundamentally the same as that of Kaufmann and Love.

It is interesting to note here that, in his papers referred to above, Das has not considered the string struck at its centre. He has, however, discussed the case of a hammer striking a string at a finite distance from one end A, and at an infinite distance from the other end B. Thus he neglects reflexions from the end B and gives the expression for pressure (P_A) between the hammer and the string at different epochs. It can be easily seen that if the hammer strikes at the centre of a finite string there will be reflexions

* Das, *loc. cit.*

from both ends, and (P_A) will be equal to (P_B). And so Das's expression for (P_A) when B is at an infinite distance from the hammer can be used to find out the pressures at any epoch when the hammer is at the centre. On integrating these expressions for pressure twice with respect to time, the corresponding expressions for the displacements can be easily obtained. On evaluating the expressions for "y" in this way, we find that during the first and the second epoch they are in agreement with those obtained by Kaufmann and Love. During the third epoch, however, the displacement found in this way is different from that of Kaufmann or Love. The acceleration during the third epoch, according to Das (*vide* P. Das, first paper), will be

$$\ddot{y} = -\frac{2\rho vc}{m}\left[e^{-\frac{2\rho ct}{m}} + 2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1-\frac{2\rho}{m}(ct-2a)\right\}\right.$$
$$\left. + 2e^{-\frac{2\rho}{m}(ct-4a)}\left\{1-\frac{4\rho}{m}(ct-4a)+\frac{2\rho^2}{m^2}(ct-4a)^2\right\}\right]. \quad (44)$$

Thus on integrating the above equation the displacement becomes

$$y = \frac{mv}{2\rho c}\left[2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1+\frac{2\rho}{m}(ct-2a)\right\} - e^{-\frac{2\rho c}{m}t} - 1\right.$$
$$\left. - \frac{4\rho^2}{m^2}(ct-4a)^2 e^{-\frac{2\rho}{m}(ct-4a)^2}\right], \quad . \quad . \quad . \quad . \quad (45)$$

which is different from the corresponding equation (8) of Kaufmann or Love. It may be remarked here, in conclusion, that neglecting $c\tau$ compared with $\frac{2\rho}{m}$, Das's equation (45) approximates to our equation (40).

Section C.

If the string is struck very near its end, the duration of contact between the hammer and the string as obtained from our equation (21) or (22), is

$$\frac{\phi}{\theta} = \frac{\pi}{\sqrt{\frac{m'}{m}\left(\frac{4l}{a}-\frac{4m'}{3m}\right)}}, \quad . \quad . \quad . \quad . \quad (46)$$

where ϕ is the duration of contact, and θ the time-period of vibration. The above equation, it may be noted, is very nearly the same as Kaufmann's expression (3).

Now if the hammer strikes at the centre of the string, the duration of contact, if the hammer leaves the string during the second epoch, can be obtained from our theory in the following way. We have from equation (36)

$$\frac{d^2y}{dT^2} = \frac{2\rho vc}{m} e^{-\frac{2\rho c}{m}\cdot T}\left(2 - e^{-\frac{4a\rho}{m}}\right). \quad . \quad . \quad . \quad (47)$$

Thus neglecting higher terms the time at which $\frac{d^2y}{dT^2} = 0$ is given by $T = \frac{m}{2\rho c}$. Thus we have

$$\frac{\phi}{\theta} = \frac{1}{2} + \frac{1}{4}\cdot\frac{m}{m'}, \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where ϕ is the duration of contact and θ the time-period of vibration of the string. It may be pointed out that the above equation (48) is the same as Kaufmann–Love's equation (7) * without the factor $1 + \frac{1}{2}e^{-\frac{2m'}{m}}$, which is nearly equal to 1. Thus the present theory gives an approximate value of the duration of contact when the hammer leaves the centre of the string during the second epoch.

Conclusion and Discussion.

In this paper we have confined ourselves to the approximate solution of the problem of the pianoforte string. The more rigorous solution by the same method on the assumption that after the sudden increase the impulses decrease continuously and exponentially, will be given in the second part of this paper. It is perhaps needless to remark here that the method of intermittent action adopted throughout this paper is far more simple than the lengthy and cumbrous method of variation of integration constant followed by Kaufmann, Love, and Das. Moreover, in view of the fact that the same method of intermittent action has found wide application in many problems of quantum physics, the present theory, which may perhaps be called the quantum theory of the pianoforte string, is of great interest even in molecular physics.

* Das's theory gives the same value of the duration of contact (second epoch) as that of Kaufmann and Love.

Appendix A.

Instead of supposing, like Kaufmann, that the shorter part of the string (a cm.) is rigid, we suppose it to have a linear density ρ_1, far greater than the linear density ρ_2 of the other part (b cm.) In that case, the frequency equation becomes (*vide* Rayleigh, 'Theory of Sound,' i.).

$$T_1\lambda_1 \frac{\cos \lambda_1 a}{\sin \lambda_1 a} + T_1\lambda_2 \frac{\cos \lambda_2 b}{\sin \lambda_2 b} = p^2 m. \quad . \quad . \quad \text{(I.)}$$

It is quite evident that if the linear density of the string is the same on the two parts, *i. e.*, if $\lambda_1 = \lambda_2$, the above equation (I.) leads to the frequency equation (10). Now on substituting $\lambda_1 = \frac{p}{c_1}$ and $\lambda_2 = \frac{p}{c_2}$ where c_1 and c_2 are the velocities of a wave along the two portions, we have from the above equation (I.)

$$\frac{T_1}{m}\left[\frac{1}{c_1}\cdot\frac{\cos \lambda_1 a}{\sin \lambda_1 a} + \frac{1}{c_2}\frac{\cos \lambda_2 b}{\sin \lambda_2 b}\right] = p. \quad . \quad . \quad \text{(II.)}$$

Again, ρ_1 being very great compared with ρ_2, c_1 will be very small compared with c_2, and hence the above equation approximates to

$$\frac{T_1}{mc_1}\cdot\frac{\cos \lambda_1 a}{\sin \lambda_1 a} = p. \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

Or, putting $p = \lambda_1 c_1$ and $T_1 = c_1^2 \rho_1$, we have

$$\frac{\rho_1}{m} = \frac{\lambda_1 \sin \lambda_1 a}{\cos \lambda_1 a}. \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

Thus we have, to a first approximation,

$$\lambda_1 = \sqrt{\frac{\rho_1}{ma}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(V.)}$$

and to a second approximation

$$\left.\begin{aligned} \frac{\rho_1}{m} &= \frac{\lambda_1\left(\lambda_1 a - \frac{\lambda_1^3 a^3}{6}\right)}{1 - \frac{\lambda_1^2 a^2}{2}} \\ &= \lambda_1^2 a\left(1 + \frac{1}{3}\lambda_1^2 a^2\right) \end{aligned}\right\}, \quad . \quad . \quad \text{(VI.)}$$

and hence, finally, we have

$$\lambda_1=\sqrt{\frac{\rho_1}{ma}-\frac{1}{3}\cdot\frac{\rho_1^2}{m^2}}. \quad . \quad . \quad . \quad \text{(VII.)}$$

Appendix B.

Let the differential equation of motion of a particle be

$$\ddot{x}+2b\dot{x}+n^2x=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

Its solution in terms of initial displacement and initial velocity is

$$x_t=e^{-bt}\left\{\dot{x}_0\frac{\sin pt}{p}+x_0\left(\cos pt+\frac{b}{p}\sin pt\right)\right\}, \quad \text{(II.)}$$

where $$p^2=n^2-b^2.$$

The solution may also be given in terms of initial displacement and initial pressure or acceleration, as

$$x_t=e^{-bt}\left\{x_0\left(\frac{b^2-p^2}{2pb}\sin pt+\cos pt\right)-\frac{1}{2pb}\ddot{x}_0\sin pt\right\}. \quad \text{(III.)}$$

Thus, if the moment is excited by only pressure or acceleration acting for a moment at time t, then for the displacement at any time t', we have from (III.)

$$x_t=-e^{-b(t-t')}\cdot\frac{1}{2pb}(\ddot{x})_{t'}\sin p(t-t'). \quad . \quad . \quad \text{(IV.)}$$

If, now, $n=0$, we have $p=ib$, and hence equation (IV.) reduces to

$$x=\frac{(\ddot{x})_{t'}}{4b^2}\left\{1-e^{-2b(t-t')}\right\}. \quad . \quad . \quad . \quad \text{(V.)}$$

Thus, if

$$b=\frac{T_1}{mc}=\frac{c\rho}{m},\ (\ddot{x})_{t'}=\frac{2\rho vc}{m},\ \text{and}\ t'=0,$$

we have

$$x_t=\frac{mv}{2\rho c}(1-e^{-\frac{2T_1}{mc}t}). \quad . \quad . \quad . \quad . \quad \text{(VI.)}$$

Physical Laboratory,
Presidency College, Calcutta, India,
Nov. 1928.

XXX. *The Theory of the Pianoforte String struck by a Hard Hammer.*—Part II. *By* Dr. K. C. Kar *and* M. Ghosh *.

IN Part I. of this paper (p. 306) we have assumed that intermittent impulses are imparted by the hammer at the centre of the string, momentarily, at regular intervals of time, each equal to $\frac{2a}{c}$, where a is half the length of the string and c the velocity of the wave along the string. On the above assumption we have derived equations for the first, second, and the third epoch, at the centre, which approximately agree with those of Kaufmann and Love. The object of the present paper is to deduce the exact equation of Kaufmann and Love, assuming that the impulses come at regular intervals of time, *i. e.* at $t=0$, $\frac{2a}{c}$, $\frac{4a}{c}$, ..., as before, but instead of acting momentarily they act suddenly and then decrease exponentially with time.

It has been shown in the previous paper that the differential equation of motion of the hammer when it strikes the string at its centre is

$$\frac{d^2y}{dt^2}+\frac{2\rho c}{m}\cdot\frac{dy}{dt}=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where m is the mass of the hammer, ρ the linear density of the string, and c the velocity of the wave along it. It has also been shown in Part I. that if an impulsive pressure $\frac{2\rho v_0 c}{m}$ acts momentarily, thus imparting a velocity v_0 to the dynamical system given by the above equation, the displacement at any time t is given by

$$y_t=\frac{v_0 m}{2\rho c}\left(1-e^{-\frac{2\rho c}{m}t}\right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is Kaufmann's equation of the hammer during the first epoch, *i. e.* $0<t<\frac{2a}{c}$. Just at the end of the first epoch $\left(t=\frac{2a}{c}\right)$, *i. e.* just at the moment when the reflected waves starting from the hammer come back to it after

* Communicated by the Authors.

reflexion from both the ends, the second impulsive pressure $\frac{4\rho v_0 c}{m}$ begins to act on the hammer. As, however, the waves produced during the first epoch give rise to the force acting on the hammer at any time during the second epoch, *i. e.* at $t=\frac{2a}{c}+\mathrm{T}$, where $0<\mathrm{T}<\frac{2a}{c}$, the pressure imparted will be $-\frac{4\rho v_0 c}{m}e^{-\frac{2\rho c}{m}.\mathrm{T}}$, being equal to $2\ddot{y}$ obtained from equation (2).

Now, if a force $-\frac{4\rho v_0 c}{m}e^{-\frac{2\rho c}{m}t'}$ acts on a dynamical system given by equation (1) at time t' for a short time dt', it will impart to it a velocity $-\frac{4\rho v_0 c}{m}e^{-\frac{2\rho c}{m}t'}\,dt'$. Substituting the above value of the velocity in equation (2), we get for the displacement at any time measured from the beginning of the second epoch,

$$y_{\mathrm{T}}=-2v_0e^{-\frac{2\rho c}{m}t'}\left\{1-e^{-\frac{2\rho c}{m}(\mathrm{T}-t')}\right\}dt'. \quad . \quad . \quad (3)$$

As, however, the force is acting continuously on the system, we have

$$y_{\mathrm{T}}=-2v_0\int_0^{\mathrm{T}}e^{-\frac{2\rho c}{m}t'}\left\{1-e^{-\frac{2\rho c}{m}(\mathrm{T}-t')}\right\}dt'. \quad . \quad (4)$$

On integrating, we get

$$y_{\mathrm{T}}=-\frac{mv_0}{2\rho c}\left[2\left(1-e^{-\frac{2\rho c}{m}.\mathrm{T}}\right)-\frac{4\rho}{m}e^{-\frac{2\rho c}{m}.\mathrm{T}}.c\mathrm{T}\right]. \quad . \quad (5)$$

On putting $c\mathrm{T}=ct-2a$, where t is the time measured from the beginning of the first epoch, we get

$$y_{\mathrm{T}}=-\frac{mv_0}{2\rho c}\left[2\left\{1-e^{-\frac{2\rho}{m}(ct-2a)}\right\}-\frac{4\rho}{m}e^{-\frac{2\rho}{m}(ct-2a)}.(ct-2a)\right]. \quad . \quad . \quad . \quad (6)$$

Thus the resultant displacement at any time during the second epoch is (from equations (2) and (6))

$$y_t=\frac{mv_0}{2\rho c}\left[\left(1-e^{-\frac{2\rho c}{m}t}\right)-2\left\{1-e^{-\frac{2\rho}{m}(ct-2a)}\right\}+\frac{4\rho}{m}e^{-\frac{2\rho}{m}(ct-2a)}.(ct-2a)\right]. \quad (7)$$

Or we have, on simplifying,

$$y_t = \frac{mv_0}{2\rho c}\left[2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1+\frac{2\rho}{m}(ct-2a)\right\} - e^{-\frac{2\rho c}{m}\cdot t} - 1\right]. \quad (8)$$

On differentiating twice with respect of time we get from equation (5)

$$\ddot{y}_T = -\frac{4\rho v c}{m} e^{-\frac{2\rho c}{m}T}\left(1-\frac{2\rho c}{m}T\right). \quad . \quad . \quad . \quad (9)$$

The above equation, therefore, gives the pressure imparted during the third epoch. Thus the displacement of the hammer during the third epoch due to the above-mentioned pressure can be obtained as before from equation (2). It is given by

$$y_T = -2v_0\int_0^T \left\{1-e^{-\frac{2\rho c}{m}(T-t')}\right\} e^{-\frac{2\rho c}{m}t'}\left(1-\frac{2\rho c}{m}t'\right)dt'. \quad (10)$$

On integrating the above equation, we get

$$y_T = -\frac{mv_0}{2\rho c}\left(\frac{2\rho c T}{m}\right)^2 e^{-\frac{2\rho c}{m}\cdot T}. \quad . \quad . \quad . \quad (11)$$

Putting $cT = ct - 4a$, where t is the time measured from the beginning of the first epoch, we get

$$y_T = -\frac{mv_0}{2\rho c}\left(\frac{2\rho}{m}\right)^2 (ct-4a)^2 e^{-\frac{2\rho}{m}(ct-4a)} \quad . \quad . \quad . \quad (12)$$

Thus the resultant displacement at any time during the third epoch is (*vide* equations (8) and (11))

$$y_t = \frac{mv}{2\rho c}\left[2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1+\frac{2\rho}{m}(ct-2a)\right\} - e^{-\frac{2\rho c}{m}t} - 1 - \frac{4\rho^2}{m^2}(ct-4a)^2 e^{-\frac{2\rho}{m}(ct-4a)}\right], \quad (13)$$

which is Das's equation for displacement calculated for the third epoch (*vide* equation (46), Part I.).

If, however, we take the pressure imparted at the beginning of the third epoch to be

$$\ddot{y}_T = -\frac{4\rho v_0 c}{m} e^{-\frac{2\rho c}{m}\cdot T}\left(1-\frac{4\rho c}{m}T\right), \quad . \quad . \quad (9')$$

we get in the same way as above Kaufmann and Love's equation for displacement during the third epoch (*vide* equation (8), Part I.).

It may be noted in conclusion that the assumption underlying the present method is that sudden impulses are given to the hammer at equal intervals of time, each equal to $\frac{2a}{c}$. These impulses are evidently given by the waves striking the hammer after reflexion from the ends. It is perhaps useless to emphasize at this stage that the idea of impulses being given intermittently does not follow from Kaufmann-Love-Das's mathematics, as Das and Datta seem to think in their recent note in Phil. Mag., Aug. 1928. Lastly, we venture to claim that our method, based on Kar's theory of intermittent action, though equally rigid, is far more simple than the tedious method of variation of integration constant followed by Kaufmann, Love, and Das.

Physical Laboratory,
Presidency College, Calcutta,
July 1929.

XXXI. *Critical Stress for Tubular Struts.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN the Supplement to your June issue Dr. Carrington (on "Critical Stresses for Tubular Struts") criticizes certain statements of mine about Southwell's formula for the collapse of tubular struts through elastic instability. There is in Dr. Carrington's paper no reference to my statement that I was dealing with "conditions which are most likely to occur in practice," or to the passage in which I pointed out that Southwell had omitted to investigate the case of failure in lobes of short wave-length. I do, however, readily admit that I ought not to have used "incorrect" in reference to Southwell's formula, but, rather, "incomplete." Moreover, I was not aware that Southwell's approximate equation for the wave-length gave such excellent results for short wave-lengths as is shown by Dr. Carrington's calculations.

May I trespass further on your space and state what I consider to be the present position of this problem? Southwell's investigation includes two types of failure: we are concerned only with the lobed type. His analysis reduces to an equation involving $\frac{p}{E}$ the ratio of collapsing

load per square inch to Young's modulus, t/a the ratio of the half-thickness to the radius, K the number of lobes, and $q = \frac{2\pi a}{\lambda}$ where $a =$ the radius and λ the wave-length of a lobe. In any practicable material $\frac{p}{E}$ is small, and therefore certain terms may be neglected. If the wave-length is long certain other terms may also be neglected, and Southwell deduced a formula for the wave-length and the collapsing load which applies in these special circumstances: in the only practical application of his work with which I was acquainted, I considered that failure with a long wave-length was impossible, and I investigated the terms in his equation which could be neglected if the wave-length was assumed to be short.

Some of the practical points which influenced me may be conveniently discussed by reference to Dr. Carrington's Tables I. and II. In the first place, a solid drawn tube with a value of $t/a = 0{\cdot}001$ is not a practicable one; so far as I know, the lightest tubes yet produced have t/a about 0·016. Dr. Carrington's particular example, 1 inch diameter and ·001 inch thick, is not a practicable tube to make (even by machining) in any length—let alone the several feet required to obtain failure in a few lobes; the lightest practicable tube, 1 inch diameter of any length, would be 1/50 inch thick. Dr. Carrington's other example in his Table II. refers to a tube having $t/R = 0{\cdot}04$: this is a practicable tube to make, but, unfortunately, no materials with which an engineer is acquainted will stand stresses of several hundreds of tons per square inch and remain elastic.

Even supposing long tubes which would fail by elastic instability could be made, I consider it almost impossible to test them so as to make them fail in two or three lobes; to do this would require that the ends should be fixed in direction, and this is a very difficult condition to fulfil. Before completing my paper I examined some methods of making a thin tube long enough to fail in a few lobes, and decided that they were quite impracticable.

The practical cases I had in mind as cases to which Southwell's results might apply were a portion of an oval tube with a large radius of curvature (*e.g.*, an interplane strut of an aeroplane) and the flanges of a spar of an aeroplane. In both of these cases there is only a relatively narrow strip of tube, held at its side either by the sharply curved ends in the case of the strut, or by the webs in the case of the spar.

Leaving out of account any consideration of practical applications, Dr. Carrington's calculations are of interest in that they show that the approximate equation which Southwell gave for the long wave-lengths also gives a remarkably good approximation for short wave-lengths.

One further point of interest may be noted. If it should ever become possible to make long tubes having such values of t/R that failure by elastic instability is possible, then Southwell's approximate equation may be used to calculate the distance of the stiffeners which will be necessary to ensure that failure shall not occur at the low values corresponding to long wave-lengths.

Yours faithfully,

ANDREW ROBERTSON.

XXXII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. viii. p. 780.]

October 23rd, 1929.—Prof. J. W. Gregory, LL.D., D.Sc., F.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Carnmenellis Granite: its Petrology, Metamorphism, and Tectonics.' By P. K. Ghosh, Ph.D., D.I.C.

The paper deals with the petrology and tectonics of the Carnmenellis granite-mass and its metamorphic aureole. The granite, which occupies an area of some 50 square miles between Falmouth and Camborne (Cornwall), was divided by the Geological Survey into (1) an earlier coarse variety and (2) a later fine variety. In this paper the coarse granite of the Survey has been subdivided into three types, which, from evidence in the field as well as in the laboratory, prove to be three distinct intrusions. The petrological characters of the granites and their differentiates are described and discussed, while twelve especially made analyses illustrate the composition of the rocks and minerals of the granitic series. Twelve analyses have also been made of the associated metamorphic rocks. These latter consist of 'greenstones', slates, and schists of various types, as well as inclusions of country-rock within the granite. Special attention has been paid to the chemical nature of the alteration-products.

The structure of the area has also been investigated, and an attempt has been made to correlate the features exhibited by the granite with those of the metamorphic aureole.

2. 'Historic Changes of Level in the Delta of the Rhone.' By Richard Dixon Oldham, F.R.S., F.G.S.

The history of the Rhone delta may be summarized as follows:—

(1) At the opening of the Pleistocene Period the whole area, together with large tracts surrounding it, was covered by a deposit of gravel and well-rounded boulders, over which the Rhone and its tributaries wandered, with no fixed bed and with a velocity of current that gave them a torrential character.

(2) A period of subsidence followed, the gradient and the speed of current were diminished, and an alluvial delta built up, which was at least as extensive as that of the present day. Two stages can be recognized in this deposit:—the lower one is prevailingly arenaceous, and small flat pebbles are found scattered through some of the beds of sand; argillaceous beds are also represented, and occasionally a stiff tenacious clay is found. The upper stage indicates a further check in the velocity of current; the deposits are of the ordinary type of deltaic alluvium, composed of fine-grained impure sand or sandy silt. Throughout both stages marine shells, of species still living, are found, and also beds of peat, the latter having been formed above, the former below sea-level.

(3) A period of uplift then set in, and the land rose, not less than 14 metres, above the level to which it had sunk, the deposits laid down were exposed to denudation, and an undulating surface of erosion was developed. The date of this uplift cannot be defined in centuries, but it must have been long prehistoric in date, for it is on the weathered surface thus carved out of the alluvium that the settlements and structures of the Romans were erected.

(4) Finally came another period of subsidence, which was not regular and continuous, but took place at intervals, alternating with others, of longer duration, during which no material change took place. One of these periods of subsidence was earlier than the establishment of Roman dominion in Provence, and, although there is no direct evidence by which it can be dated, analogy with later events suggests that it probably took place between the years 1000 and 1500 B.C. The next change took place in the course of the 8th and 9th centuries; it amounted to about 5 metres of vertical displacement, and resulted in great geographical changes both at the time and during the succeeding centuries. Large areas of what had been dry land were submerged, and the silting-up of these spreads of shallow water led to a rapid extension of the land, and repeated changes in the course of the river. Finally, there was a fresh movement of subsidence, of shorter duration and smaller amount, for it seems to have been practically

completed during the latter half of the 18th century, and to have amounted to little more than a metre, if so much. The total amount of these movements of subsidence was about 10 metres, and at the end of them the land still stands about 4 metres, or more, above the lowest level reached before the period of uplift set in.

November 6th, 1929.—Prof. J. W. Gregory, LL.D., D.Sc., F.R.S., President, in the Chair.

The following communication was read :—

'On the Glaciation of Clun Forest, Radnor Forest, and some Adjoining Districts.' By Major Arthur Richard Dwerryhouse, T.D., D.Sc., M.R.I.A., F.G.S., and A. Austin Miller, M.Sc.

The district described comprises the county of Radnor and parts of Montgomeryshire, Shropshire, Brecknockshire, Herefordshire, and Worcestershire, and is some 1000 square miles in area.

The ice, derived originally from the highlands of Central Wales, filled first the depression now occupied by the valleys of the Rivers Ithon and Irfon to the west of the great line of escarpment extending from Kerry Hill on the north, by Radnor Forest and Aberedw Hill, to Mynydd Epynt on the south side of the Wye Gorge.

As the ice accumulated in the depression, it first found escape by the valleys of the Severn and the Mule on the north, and by the Wye Gorge on the south.

Gradually the level of the ice rose, until it overtopped the escarpment throughout its whole length, with, perhaps, the exception of the highest parts of Radnor Forest.

The courses of the various glaciers which were thus formed in the valleys of the dip-slope are traced, and their effects, both temporary and permanent, on the drainage of the area are discussed.

The eastern boundary of the Welsh ice has been traced from Chirbury to Hereford, and the date of its culmination is shown from evidence collected in the neighbourhood of Marrington Dingle to have been subsequent to the maximum extension of the Irish-Sea Ice on the Shropshire Plain, and to belong to the period of Lake Lapworth, but not to that of the earlier and more elevated Lake Buildwas.

The principal permanent deflexion of drainage described is that of the River Teme from its pre-Glacial course from Leintwardine by way of Wigmore, Aymestry, Mortimer's Cross, and Weobley to the Wye, into its present channel by way of Downton and Ludlow gorges to Wooferton, and thence by Tenbury across the Malvern Axis to the Severn.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

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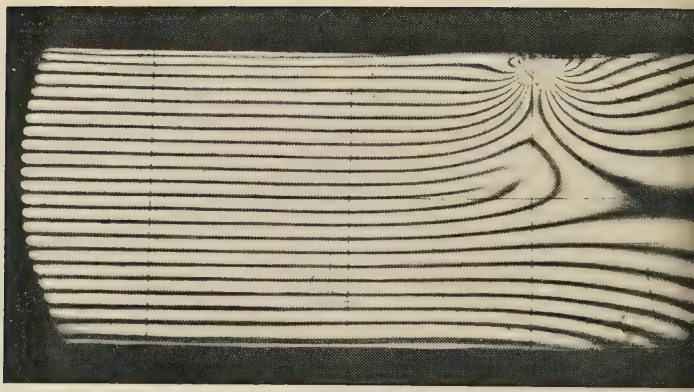

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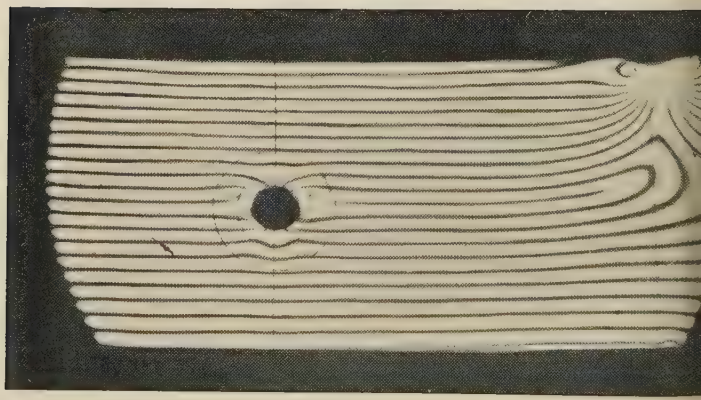

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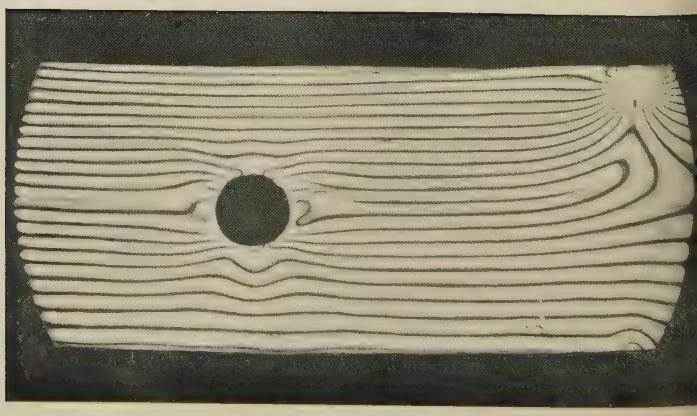

d.

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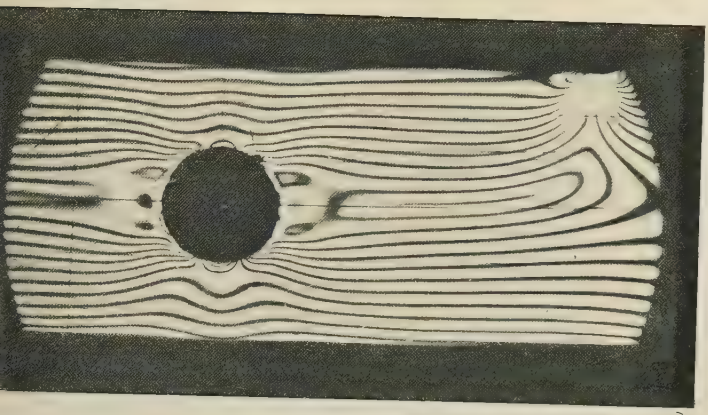

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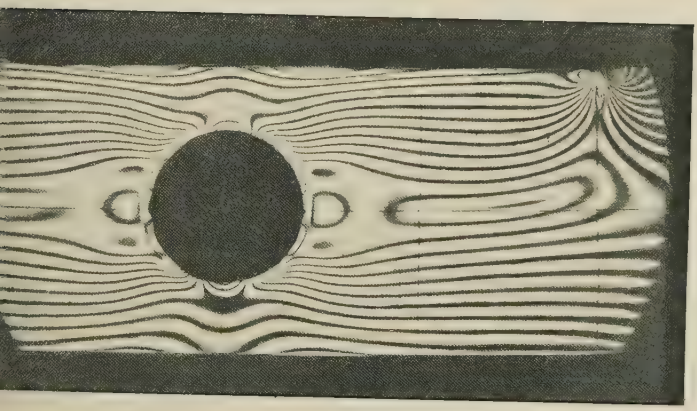

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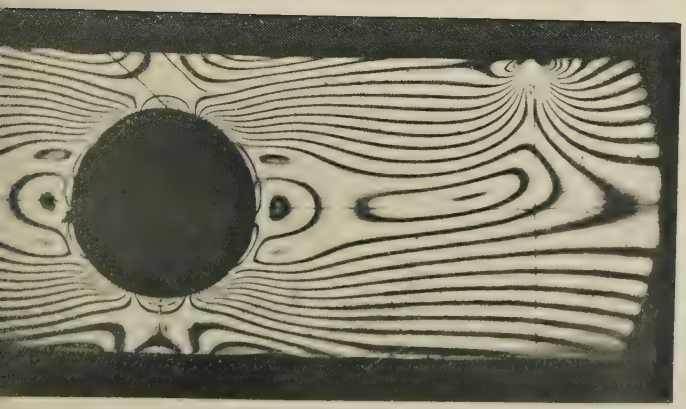

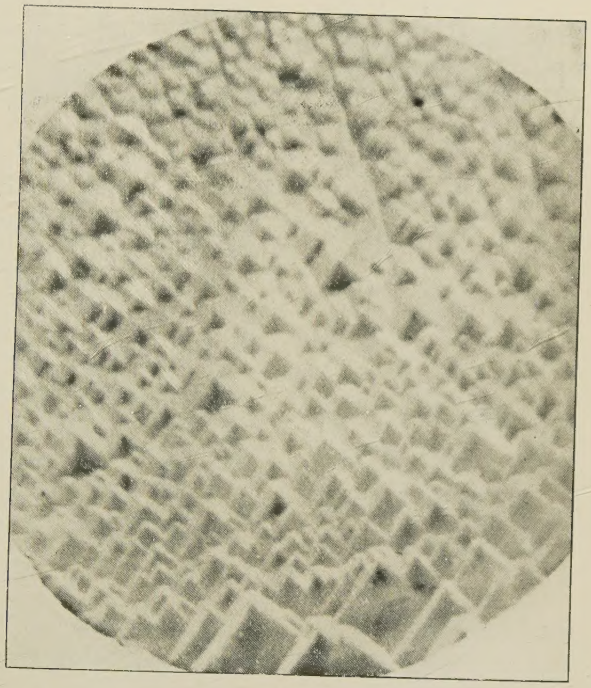

Etch-figures of Sylvine.

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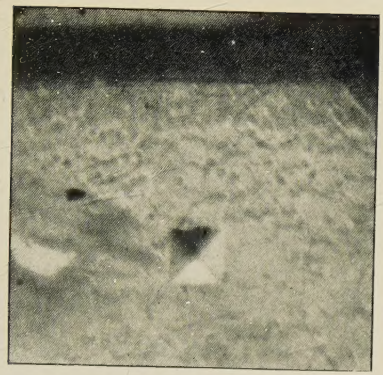

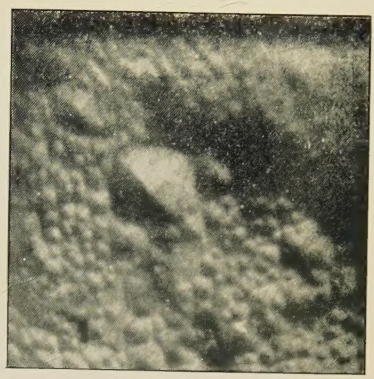

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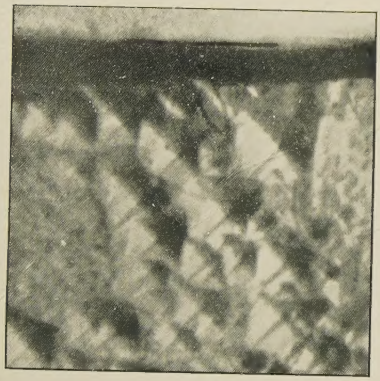
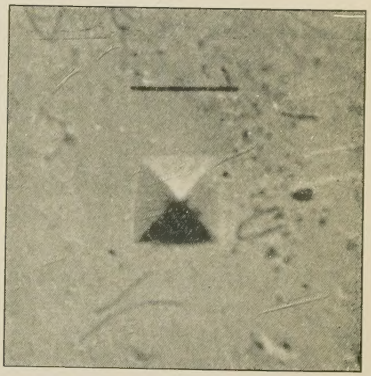
(c)





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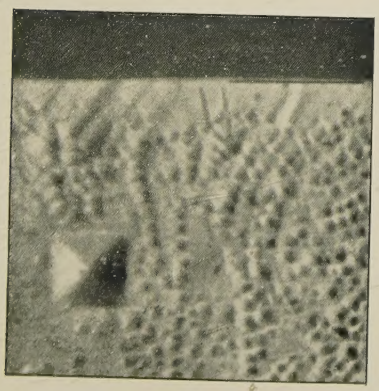

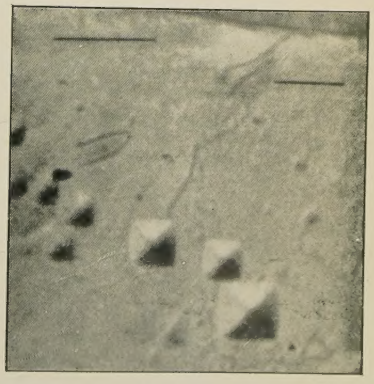

(f)

Etch-figures of Sylvine.